6.3: Working with Normally Distributed Variables

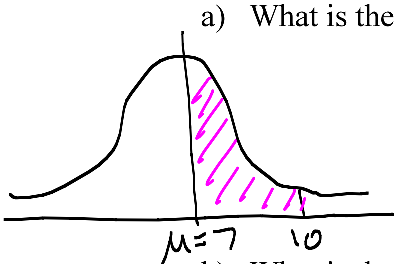
Recall: The z-score of a data point is its distance from the mean, measured in standard deviations.

> Standardizing the values of a normal distribution:
>
> In a normal distribution with mean μ and standard deviation σ, where x is a data value, the z-score is
>
> $$z = \frac{x-\mu}{\sigma}.$$
>
> The area under a normal curve between $x = a$ and $x = b$ is the same as the area under the standard normal curve between the z-score for a and the z-score for b.

Example 1: Consider a normal curve with mean 7 and standard deviation 2.

$\mu = 7, \ \sigma = 2$

a) What is the area under the curve between 7 and 10?



Find z-score corresponding to $x=10$:

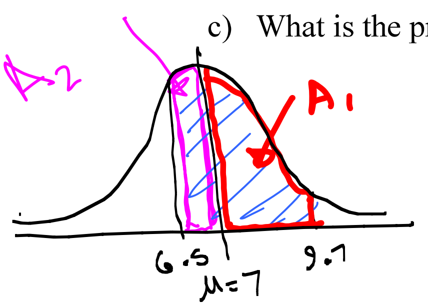
$$z = \frac{x-\mu}{\sigma} = \frac{10-7}{2} = \frac{3}{2} = 1.50$$

From table, $z=1.50$ corresponds to an $\boxed{\text{Area} = 0.4332}$

b) What is the probability that the variable is between 7 and 10?

$$P(7 < X < 10) = P(0 < Z < 1.50) = \boxed{0.4332}$$

c) What is the probability that the variable is between 6.5 and 9.7?



z-score for 9.7 is $z = \frac{x-\mu}{\sigma} = \frac{9.7-7}{2} = 1.35$

From table, $A_1 = 0.4115$

z-score for 6.5 is $z = \frac{x-\mu}{\sigma} = \frac{6.5-7}{2} = -0.25$

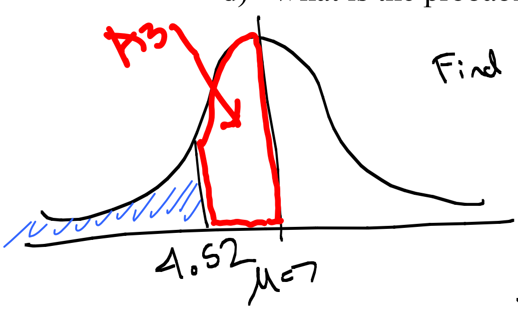
From table, $A_2 = 0.0987$

$P(6.5 < X < 9.7) = P(-0.25 < Z < 1.35) = A_1 + A_2$

$= 0.0987 + 0.4115$

$= \boxed{0.5102}$

d) What is the probability that the variable is less than 4.52?



Find z-score for $X = 4.52$:

$$z = \frac{x-\mu}{\sigma} = \frac{4.52-7}{2} = -1.24$$

From table, for $z = -1.24$, $A_3 = 0.3925$

$P(X < 4.52) = P(Z < -1.24) = 0.5 - A_3 = 0.5 - 0.3925$

$= \boxed{0.1075}$

<u>Properties of Normal Probability Distributions</u>:

1. $P(a \le x \le b) =$ area under the curve from a to b.
2. $P(-\infty \le x \le \infty) = 1 =$ total area under the curve.
3. $P(x = c) = 0$.

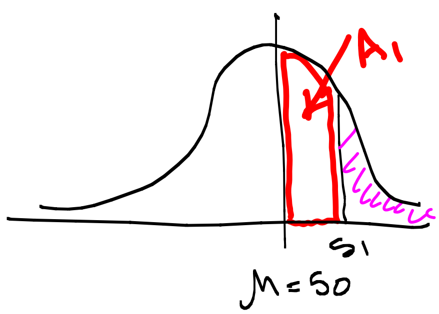
Note: $P(a \le x \le b) = P(a \le x < b) = P(a < x \le b) = P(a < x < b)$

Example 2: Dusty Dog Food Company ships dog food to its distributors in bags whose weights are normally distributed with a mean weight of 50 pounds and standard deviation 0.5 pound. If a bag of dog food is selected at random from a shipment, what is the probability that it weighs

$\mu = 50, \quad \sigma = 0.5$

a) More than 51 pounds?
b) Less than 49 pounds?
c) Between 49 and 51 pounds?
d) What is the percentage of dog food bags that weigh more than 51 pounds?

(a)



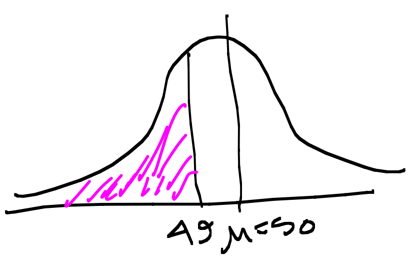
Z-score for 51 is

$$z = \frac{x-\mu}{\sigma} = \frac{51-50}{0.5} = 2.00$$

From table, $A_1 = 0.4772$

$$P(x > 51) = 0.5 - 0.4772 = \boxed{0.0228}$$

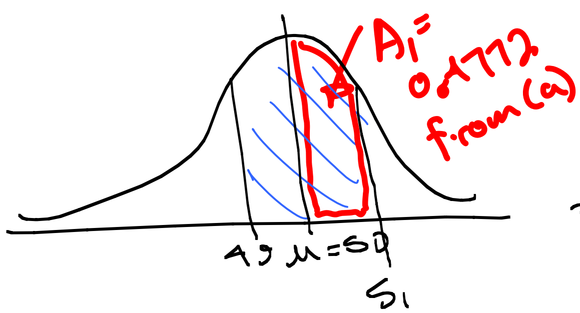
(b)



From symmetry, this is the same area as in part (a).

$$P(x < 49) = \boxed{0.0228}$$

(c)



From symmetry,

$$P(49 < x < 51) = 2(0.4772)$$
$$= \boxed{0.9544}$$

(d) From part (a) $P(x > 51) = 0.0228$.

So about $\boxed{\text{2.28\% of bags weigh more than 51 lbs.}}$

Example 3: The medical records of infants delivered at a certain hospital show that the infants' birth weights in pounds are normally distributed with a mean of 7.4 and a standard deviation of 1.2.

a) What percentage of infants at this hospital weighed more than 9.2 pounds at birth?
b) What percentage of infants at this hospital weighed less than 8 pounds at birth?
c) What percentage of infants at this hospital weighed between 8 and 10 pounds at birth?

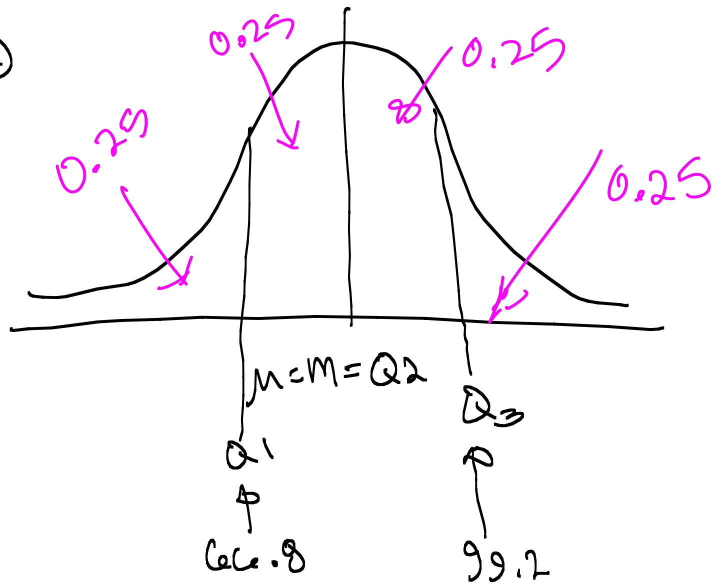
Important: The *z*-score is the number of standard deviations between the data point and the mean.

Example 4: A variable is normally distributed with mean 83 and standard deviation 24.

$\mu = 83, \quad \sigma = 24$

a) Find and interpret the quartiles.
b) Find and interpret the 98th percentile.
c) Find and interpret the first and second deciles.
d) Find the value that 72% of all possible values of the variable exceed.
e) Find two values of the variable that divide the area into a middle area of 0.90 and two outside areas of 0.05 each.

a



So Q_3 corresponds to $z_{0.25}$

Look up Area = 0.25 in table

It's between A = 0.2486 and A = 0.2517, corresponding to z = 0.67 and z = 0.68

so $z_{0.25} \approx 0.675$

$$\boxed{\begin{aligned} Q_1 &= 66.8 \\ Q_2 &= \mu = \text{median} = 83 \\ Q_3 &= 99.2 \end{aligned}}$$

(25% of data lie below 66.8, 25% of data between 66.8 and 83, 25% between 83 and 99.2, and 25% above 99.2)

Use $z = \dfrac{x-\mu}{\sigma}$

$$0.675 = \frac{x-83}{24}$$

$$24(0.675) = x - 83$$

$$24(0.675) + 83 = x$$

$$99.2 = x$$

So $Q_3 = 99.2$

For Q_1, we use $z = -0.675$ (symmetry)

so $x = 83 - 0.675(24)$
$= 66.8$

$Q_1 = 66.8$

Note: solve $z = \dfrac{x-\mu}{\sigma}$ for x:

$$z\sigma = x - \mu$$

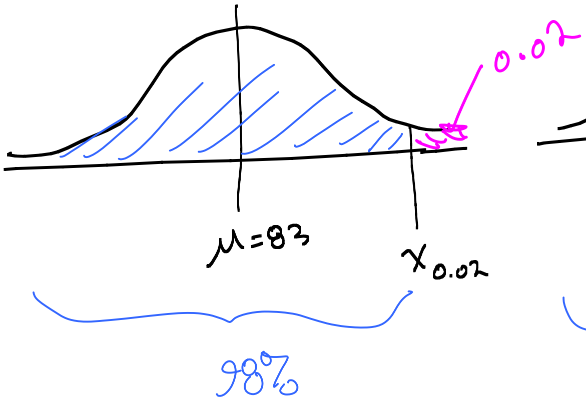
$$z\sigma + \mu = x$$

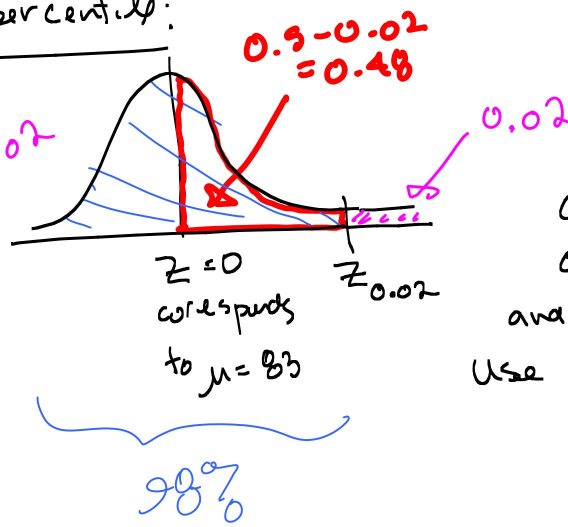
$$x = \mu + z\sigma$$

so, for $z = 0.675$ at Q_3, we get $x = 83 + 0.675(24)$
$= 99.2$

Example 4 cont'd:

ⓑ Find the 98th percentile:





Look up
Area = 0.48
in Table.

Closest areas are
0.4798 (with z = 2.05)
and 0.4803 (with z = 2.06)

Use z = 2.055

we have $z = 2.055$

$$z = \frac{x-\mu}{\sigma}$$

$$z\sigma = x - \mu$$

$$\mu + z\sigma = x$$

$$x = \mu + z\sigma$$

Given: $\mu = 83$
$\sigma = 24$

$x = 83 + 2.055\,(24) = \boxed{132.32}$ is the 98th percentile.

(This means that about 98% of data points are below 132.32 and 2% are above).

6.3.4c

c) Find 1st and 2nd deciles

Quartiles: break data set into 4 parts (same number of data pts in each part).

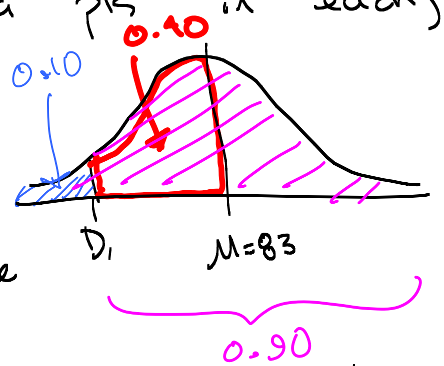
Deciles: Break data set into 10 parts (same # of data pts in each)

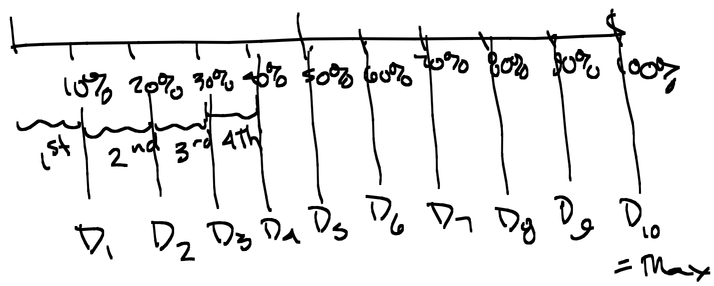
Find 1st decile

1st decile: D_1

10% are below

90% are above





Look up Area = 0.40 in table. Closest area is 0.3997, corresponding to $z = 1.28$.

D_1 is left of the mean, so use $z = -1.28$.

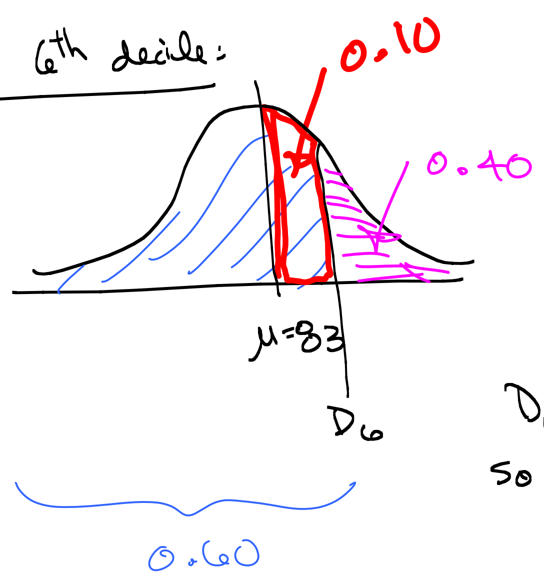
$x = \mu + z\sigma$

$x = 83 - 1.28(24) = 52.28$

$\boxed{\text{1st decile is } 52.28.}$

For D_6: 60% of pts will be below D_6; 40% above

Find 6th decile:



Look up Area = 0.10 in table:

Closest area is 0.0987, corresponding to $z = 0.25$

D_6 is to the right of the mean, so use the positive z-score.

$x = \mu + z\sigma = 83 + 0.25(24) = 89$

$\boxed{\text{6th decile is } 89}$

Note:

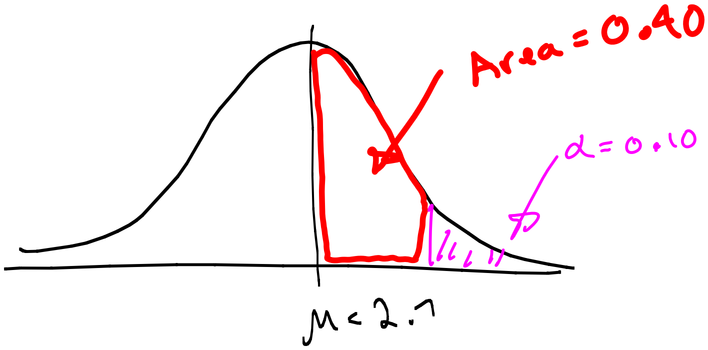
1st decile: 10th percentile

2nd decile: 20th percentile

⋮

6th decile 60th percentile

Example 5: The GPA of the senior class of a certain high school is normally distributed with a mean of 2.7 and a standard deviation of 0.4 point. If a senior in the top 10% of his or her class is eligible for admission to any state university, what is the minimum GPA that a senior should have to ensure eligibility to a state university?



Look up area = 0.40 in table.