1.3: Evaluating Limits Analytically

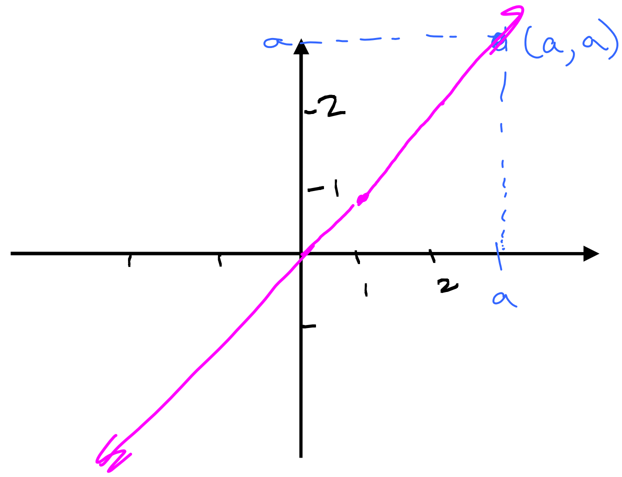
Some basic limits:

Example 1: Determine $\lim_{x \to a} x$.

Here, $f(x) = x$

Let's graph $y = x$.

$$\boxed{\lim_{x \to a} x = a}$$

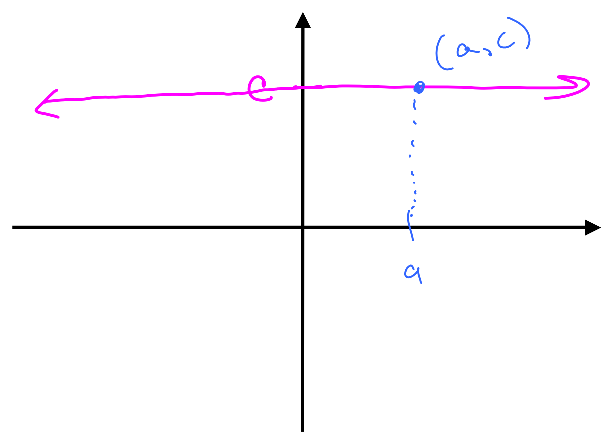


Example 2: Determine $\lim_{x \to a} c$.

Graph $y = c$.

Here, $f(x) = c$

$$\boxed{\lim_{x \to a} c = c}$$



Laws (or properties) of limits:

<u>Limit Laws</u>:

Suppose that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ if c is a constant

4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a}[f(x)]^{n/p} = [\lim_{x\to a} f(x)]^{n/p}$, if n and p are integers with no common factor, $p \neq 0$, and provided that $[\lim_{x\to a} f(x)]^{n/p}$ is a real number.

Combining the facts that $\lim_{x\to a} c = c$, $\lim_{x\to a} x = a$, and the limit laws shows that $\lim_{x\to a} x^n = a^n$ where n is a positive integer. This lets us use direct substitution for evaluating limits of polynomials.

<u>Direct Substitution Property</u>:

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

<u>Example 3:</u> Determine $\lim_{x\to 3}(4x^2 - 2x + 1)$.

$\lim_{x\to 3}(4x^2 - 2x + 1) = 4(3)^2 - 2(3) + 1 = 36 - 6 + 1 = \boxed{31}$

<u>Example 4:</u> Determine $\lim_{x\to -2}\dfrac{2x^2-6x+5}{x-3}$.

$\boxed{-5}$

Example 5: Determine $\lim_{x \to 2} \sqrt[3]{4x - x^4}$.

$$\lim_{x \to 2} \sqrt[3]{4x - x^4} = \sqrt[3]{4(2) - 2^4} = \sqrt[3]{8-16} = \sqrt[3]{-8} = \boxed{-2}$$

Note: $(-2)^3 = (-2)(-2)(-2) = -8$

Limit Law #6 (used in the previous example) is a special case of the following theorem:

Limit of a Composite Function If f and g are functions such that $\lim_{x \to a} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$ $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L)$

Example 6: Determine $\lim_{x \to 4} \dfrac{x^2 + x - 20}{x^2 - 7x + 12}$.

If we try direct substitution, we get $\frac{0}{0}$, which is not defined. This is an example of a $\frac{0}{0}$ indeterminate forms

$$\lim_{x \to 4} \frac{x^2 + x - 20}{x^2 - 7x + 12}$$

$$= \lim_{x \to 4} \frac{(x+5)\cancel{(x-4)}}{\cancel{(x-4)}(x-3)} = \lim_{x \to 4} \frac{x+5}{x-3} = \frac{4+5}{4-3}$$

$$= \frac{9}{1} = \boxed{9}$$

Example 7: Determine $\lim_{x \to -2} \dfrac{\sqrt{x+3} - 1}{x+2}$.

Direct substitution $\Rightarrow$

$$\frac{\sqrt{-2+3} - 1}{-2+2} \Rightarrow \frac{0}{0}$$

indeterminate

$$\lim_{x \to -2} \frac{\sqrt{x+3} - 1}{x+2} = \lim_{x \to -2} \frac{\sqrt{x+3} - 1}{x+2} \left(\frac{\sqrt{x+3} + 1}{\sqrt{x+3} + 1} \right)$$

$$= \lim_{x \to -2} \frac{(\sqrt{x+3})^2 - 1^2}{(x+2)(\sqrt{x+3} + 1)} = \lim_{x \to -2} \frac{x+3-1}{(x+2)(\sqrt{x+3} + 1)}$$

$$= \lim_{x \to -2} \frac{\cancel{x+2}}{\cancel{(x+2)}(\sqrt{x+3} + 1)} = \lim_{x \to -2} \frac{1}{\sqrt{x+3} + 1} = \frac{1}{\sqrt{-2+3} + 1} = \frac{1}{\sqrt{1} + 1}$$

$$= \boxed{\frac{1}{2}}$$

Define absolute value:

If $f(x) = |x|$, how is f defined?

$$|x| = f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

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$$\lim_{x \to 0} \frac{|x+1| - |x-1|}{x}$$

Let $f(x) = \dfrac{|x+1| - |x-1|}{x}$

For $x > 0$, near 0, $f(x) = \dfrac{x+1-(-(x-1))}{x} = \dfrac{x+1-(-x+1)}{x}$

$$= \frac{x+1+x-1}{x} = \frac{2x}{x} = 2$$

f $x < 0$ near 0 $f(x) = \dfrac{x+1-(-(x-1))}{x} = \dfrac{x+1+x-1}{x} = \dfrac{2x}{x} = 2$

So $\lim_{x \to 0} f(x) = \boxed{2}$

Example 8: Determine $\lim_{x\to 3}\frac{x+4}{x-3}$.

Skip for now

Limits of trigonometric functions:

These can also be evaluated through direct substitution, thanks to the two limits below, along with the limit laws.

$$\lim_{x\to a}\sin x = \sin a \qquad \lim_{x\to a}\cos x = \cos a$$

Example 9: Determine $\lim_{x\to \frac{\pi}{3}}\tan x$.

$$\lim_{x\to \frac{\pi}{3}} \tan(x) = \lim_{x\to\frac{\pi}{3}} \frac{\sin(x)}{\cos(x)}$$

\

$$= \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{2}\cdot\frac{2}{1} = \frac{\sqrt{3}}{1} = \boxed{\sqrt{3}}$$

Example 10: Determine $\lim_{x\to \frac{\pi}{4}}\frac{\cos x - \sin x}{\cos(2x)}$

Direct substitution:

$$\frac{\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{2\pi}{4}\right)}$$

$$\frac{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}{0} \qquad \frac{0}{0}$$

indeterminate form

$$\lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos^2 x - \sin^2 x}$$

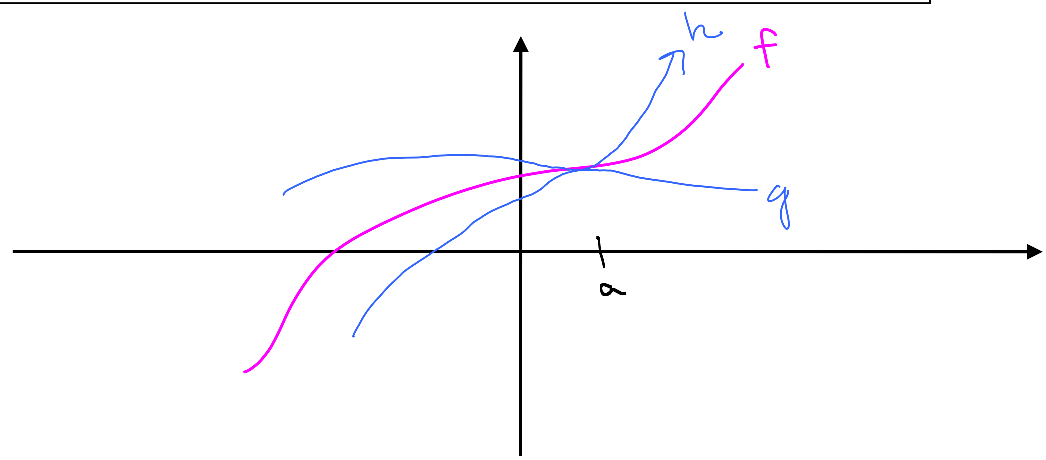
$$= \lim_{x\to\frac{\pi}{4}} \frac{\cancel{\cos x - \sin x}}{(\cos x + \sin x)\cancel{(\cos x - \sin x)}}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{1}{\cos x + \sin x} = \frac{1}{\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)} = \frac{1}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}} = \frac{1}{\frac{\cancel{2}\sqrt{2}}{\cancel{2}}} = \boxed{\frac{1}{\sqrt{2}}} = \boxed{\frac{\sqrt{2}}{2}}$$

The Squeeze (or Sandwich or Pinching) Theorem:

If $g(x) \le f(x) \le h(x)$ for all x in some open interval containing near a, except possibly at a itself.
If $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then

$$\cancel{\lim_{x \to a} h(x) = L}. \qquad \lim_{x \to a} f(x) = L.$$



Example 11: Use the Squeeze Theorem to show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

skip

Two important limits:

Know these

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$$

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)=0$$

Note: This means that $\lim_{x\to 0}\left(\frac{x}{\sin x}\right)=1$, $\lim_{x\to 0}\left(\frac{\cos x-1}{x}\right)=0$, and $\lim_{x\to 0}\left(\frac{x}{1-\cos x}\right)$ does not exist.

$$\lim_{x\to 0}\left(\frac{x}{\sin x}\right)=\lim_{x\to 0}\left(\frac{1}{\frac{\sin x}{x}}\right)=\frac{1}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)}=\frac{1}{1}=\boxed{1}$$

Example 12: $\lim_{x\to 0}\left(\frac{\cos x\tan x}{x}\right)$

$$\lim_{x\to 0}\left(\frac{\cos x\tan x}{x}\right)=\lim_{x\to 0}\left(\frac{\cancel{\cos x}\left(\frac{\sin x}{\cancel{\cos x}}\right)}{x}\right)=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=\boxed{1}$$

Example 13: $\lim_{x\to \frac{\pi}{2}}\left(\frac{\cos x}{\cot x}\right)$ Work out using direct substitution

Example 14: $\lim_{x\to 0}\left(\frac{1-\cos(5x)}{x}\right)$

$$\lim_{x\to 0}\left(\frac{1-\cos(5x)}{x}\right)=\lim_{x\to 0}\left(\frac{1-\cos(5x)}{x}\right)\left(\frac{5}{5}\right)=\frac{5}{1}\lim_{x\to 0}\left(\frac{1-\cos(5x)}{5x}\right)$$

$$=5\lim_{5x\to 0}\left(\frac{1-\cos(5x)}{5x}\right)=\boxed{0}$$

Note: as $x\to 0$, $5x\to 0$ also
$\lim_{x\to 0}(5x)=5(0)=0$

Example 15: $\lim\limits_{x\to 0}\left(\dfrac{\sin(3x)}{7x}\right)$

$$\lim_{x\to 0}\left(\frac{\sin(3x)}{7x}\right) = \frac{1}{7}\lim_{x\to 0}\left(\frac{\sin(3x)}{x}\right)\left(\frac{3}{3}\right) = \frac{1}{7}\cdot\frac{3}{1}\lim_{x\to 0}\left(\frac{\sin(3x)}{3x}\right)$$

$$= \frac{3}{7}\cdot\lim_{3x\to 0}\left(\frac{\sin(3x)}{3x}\right) = \frac{3}{7}(1) = \boxed{\frac{3}{7}}$$

Note: as $x\to 0$, $3x\to 0$ also

Example 16: $\lim\limits_{x\to 0}\dfrac{\sin(3x)}{\sin(2x)}$

$$\lim_{x\to 0}\frac{\sin(3x)}{\sin(2x)} = \lim_{x\to 0}\left[\frac{\sin(3x)}{\sin(2x)}\cdot\frac{x}{x}\right] = \lim_{x\to 0}\left[\frac{\sin(3x)}{x}\cdot\frac{x}{\sin(2x)}\right]\left(\frac{2}{2}\right)\left(\frac{3}{3}\right)$$

$$= \frac{3}{1}\cdot\frac{1}{2}\cdot\lim_{x\to 0}\left[\frac{\sin(3x)}{3x}\cdot\frac{2x}{\sin(2x)}\right] = \frac{3}{2}\left[\lim_{x\to 0}\left(\frac{\sin(3x)}{3x}\right)\lim_{x\to 0}\left(\frac{2x}{\sin(2x)}\right)\right]$$

$$= \frac{3}{2}\left[\lim_{3x\to 0}\left(\frac{\sin(3x)}{3x}\right)\lim_{2x\to 0}\left(\frac{2x}{\sin(2x)}\right)\right]$$

$$= \frac{3}{2}(1)(1) = \boxed{\frac{3}{2}}$$

Note: as $x\to 0$, $3x\to 0$ also.
as $x\to 0$, $2x\to 0$ also.

Example 17: $\lim\limits_{x\to 0}\dfrac{x^3}{\sin^3(4x)}$

$$\lim_{x\to 0}\left(\frac{x}{\sin(4x)}\right)^3$$

Example 18: $\lim\limits_{x\to 0}\dfrac{\cot 3x}{\csc 8x}$