1.4: Continuity and One-Sided Limits

One-Sided Limits:

$\lim_{x \to a^-} f(x) = L$ means that $f(x)$ approaches L as x approaches a from the left.

$\lim_{x \to a^-} f(x) = L$

$\lim_{x \to a^+} f(x) = L$ means that $f(x)$ approaches L as x approaches a from the right.

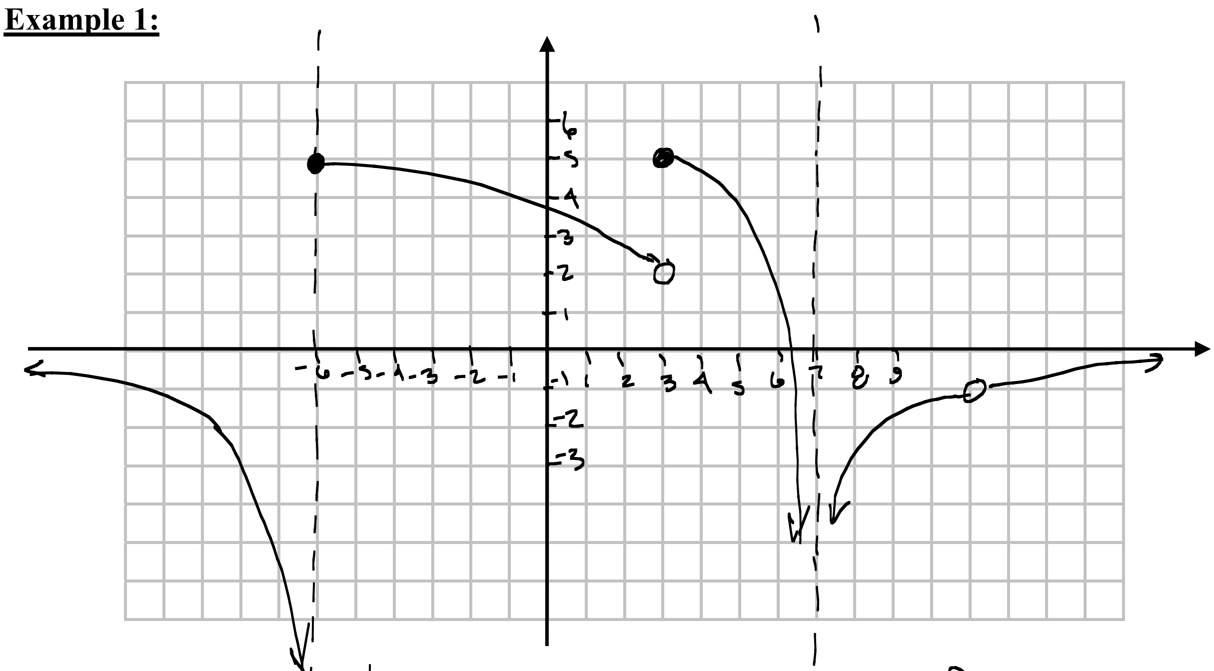
$\lim_{x \to a} f(x) = L$

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

$\lim_{x \to a} f(x) = L$ if and only if (iff) $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Example 1:

f



$\lim_{x \to -6^+} f(x) = \underline{5}$

$\lim_{x \to -6^-} f(x) = \underline{-\infty}$ (does not exist)

$\lim_{x \to 3^-} f(x) = \underline{2}$

$\lim_{x \to 3^+} f(x) = \underline{5}$

$\lim_{x \to 3} f(x)$ does not exist

(left-hand and right-hand limits do not match)

$\lim_{x \to 11^+} f(x) = \underline{-1}$

$\lim_{x \to 11^-} f(x) = \underline{-1}$

So $\lim_{x \to 11} f(x) = -1$

Example 2: Determine $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, and $\lim_{x\to 1} f(x)$. also $\lim_{x\to 2} f(x)$

$$f(x)=\begin{cases} x^2 & \text{if } x\le 1 \\ x-3 & \text{if } x>1 \end{cases}.$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x-3) = 2-3 = \boxed{-1}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2) = 1^2 = \boxed{1}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x-3) = 1-3 = \boxed{-2}$$

$(=1)$ $(=-2)$

$\lim_{x\to 1} f(x)$ does not exist, because $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

Example 3: Determine $\lim_{x\to -2^-} f(x)$, $\lim_{x\to -2^+} f(x)$, and $\lim_{x\to -2} f(x)$.

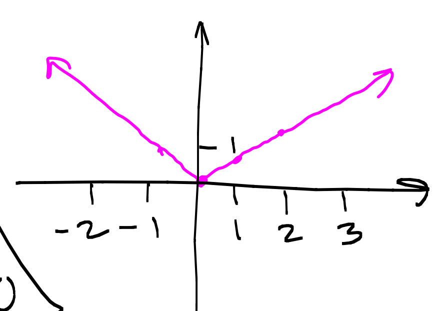
$$f(x)=\begin{cases} 1-x^3 & \text{if } x\le -2 \\ 7-x & \text{if } x>-2 \end{cases}.$$

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-}(1-x^3) = 1-(-2)^3 = 1-(-8) = 1+8 = \boxed{9}$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+}(7-x) = 7-(-2) = 7+2 = \boxed{9}$$

$$\lim_{x\to -2} f(x) = \boxed{9}$$

$f(x) = |x|$



Example 4: Determine $\lim_{x\to 0}|x|$, if it exists.

From graph, $\lim_{x\to 0}|x| = \boxed{0}$

OR

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x<0 \end{cases}$$

so $\lim_{x\to 0}|x| = 0$

Now, $\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-x) = -0 = 0$ AND $\lim_{x\to 0^+}|x| = \lim_{x\to 0^+}(x) = 0$

Example 5: Determine $\lim_{x\to 1} f(x)$ and $\lim_{x\to -2} f(x)$, where $f(x)=\begin{cases}\sqrt{5-x} & \text{if } x\geq 1\\ 3x^2-7 & \text{if } x<1\end{cases}$.

Example 6: Determine $\lim_{x\to 0} f(x)$, where $f(x)=\begin{cases}\cos x & \text{if } x<0\\ 2 & \text{if } x=0\\ \sqrt{x+1} & \text{if } x>0\end{cases}$.

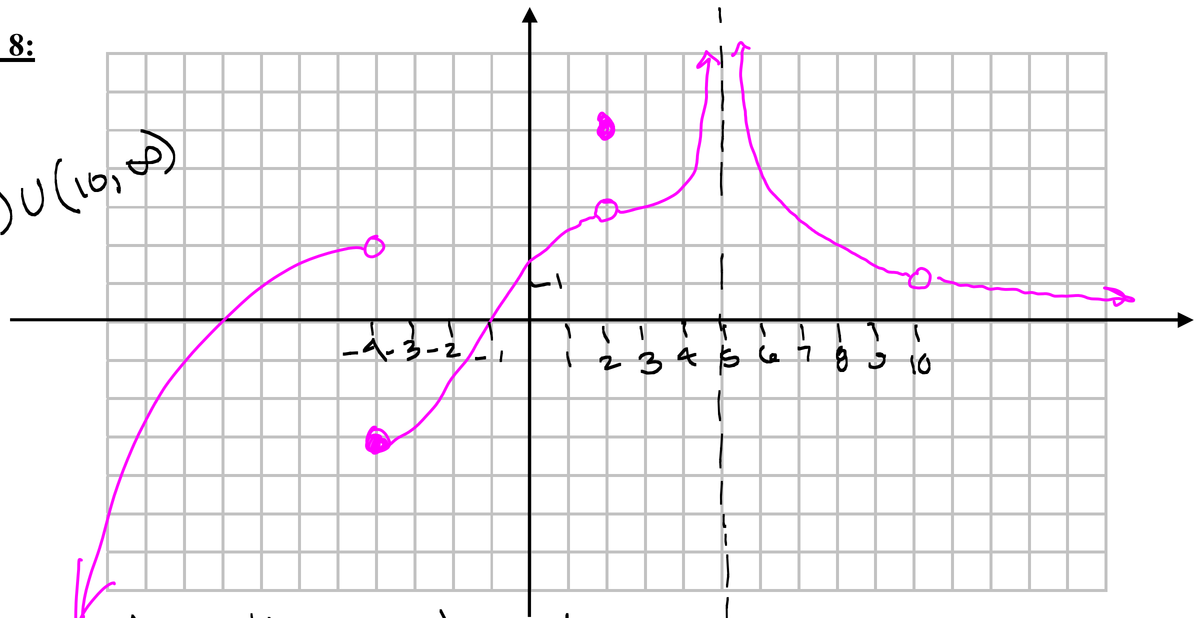
Example 7: Determine $\lim_{x\to 1}\frac{|x-1|}{x-1}$.

Continuity of a function:

In most cases, we can think of a continuous function as one that can be drawn “without lifting your pencil from the paper”. In other words, there are no holes, breaks, or jumps.

Example 8:

Domain: $(-\infty, 5) \cup (5, 10) \cup (10, \infty)$



This function is discontinuous at $-4, 2, 5, 10$

Continuous on each of the intervals: $(-\infty, -4)$, $(-4, 2)$, $(2, 5)$, $(5, 10)$, $(10, \infty)$

Definition: A function f is continuous at a number a, an interior point of its domain, if

$$\lim_{x \to a} f(x) = f(a)$$

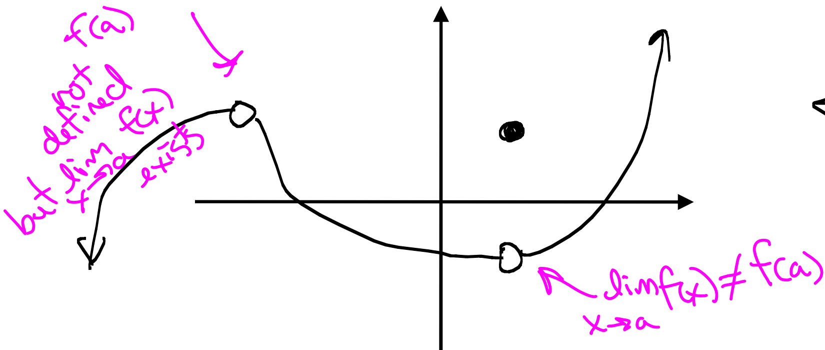
Conditions for Continuity:

In order for f to be continuous at a, all the following conditions must hold:

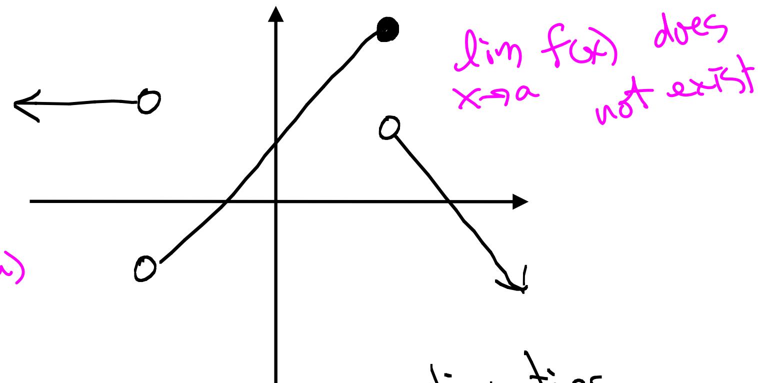
1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Types of discontinuities:
1. Removable discontinuity 2. Infinite discontinuity 3. Jump discontinuity 4. Oscillating discontinuity

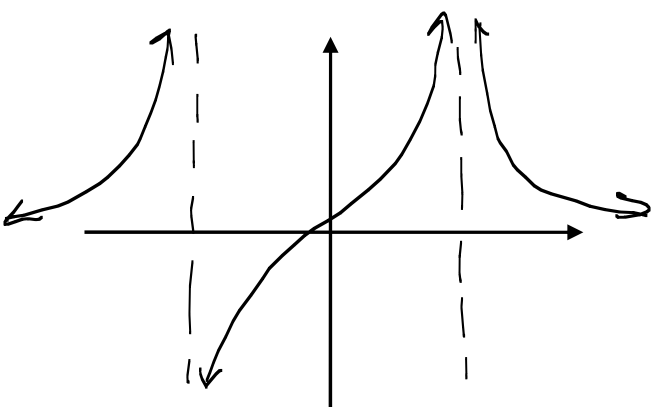
What do these look like? For each type of discontinuity, what condition of continuity is violated?



Removable Discontinuities
(could be removed by modifying
y-value)



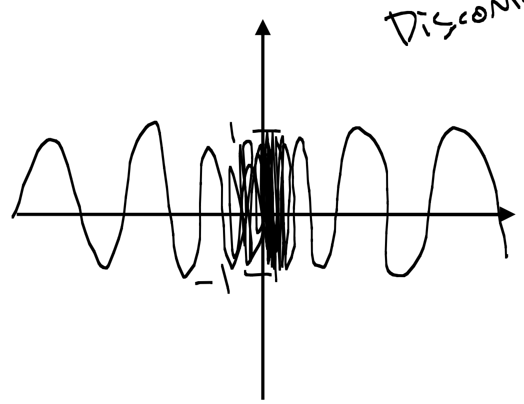
Jump discontinuities
(sides don't match)



Infinite discontinuities

$\lim_{x \to a} f(x)$ does not exist

$y = \sin\left(\frac{1}{x}\right)$
Discontinuous at 0



Oscillating Discontinuity

$\lim_{x \to a} f(x)$ does not exist

Continuity at an endpoint of the domain:

A function f is <u>continuous at a number *a*</u>, a left endpoint of its domain, if $\lim_{x \to a^+} f(x) = f(a)$.

A function f is <u>continuous at a number *a*</u>, a right endpoint of its domain, if $\lim_{x \to a^-} f(x) = f(a)$.

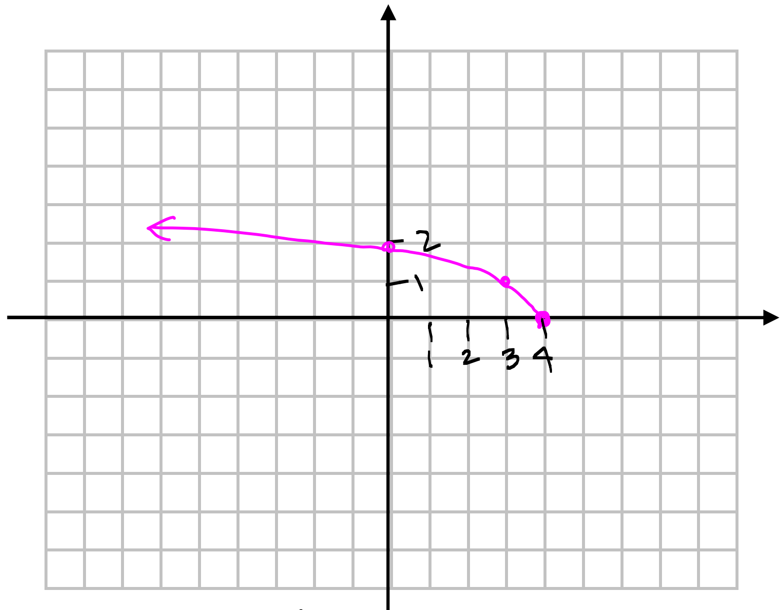
<u>Example 9:</u> $f(x) = \sqrt{4 - x}$

$4 - x \geq 0$

$4 \geq x$

$x \leq 4$

Domain: $(-\infty, 4]$



This function is continuous at 4 because $\lim_{x \to 4^-} f(x) = f(4) = 0$

f is continuous on $(-\infty, 4]$

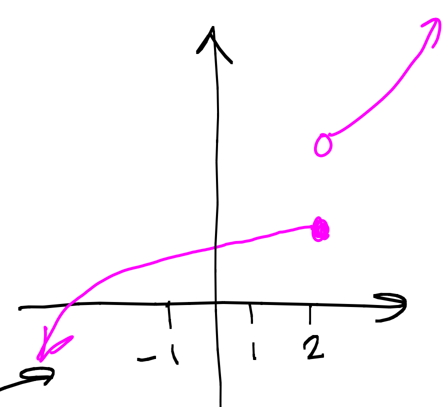
One-sided continuity:

<u>Definition</u>: A function f is continuous from the left at a number a if

$$\lim_{x \to a^-} f(x) = f(a) \qquad \lim_{x \to a^-} f(x) = f(a)$$

A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \lim_{x \to a^+} f(x) = f(a)$$



f is continuous from the left at 2

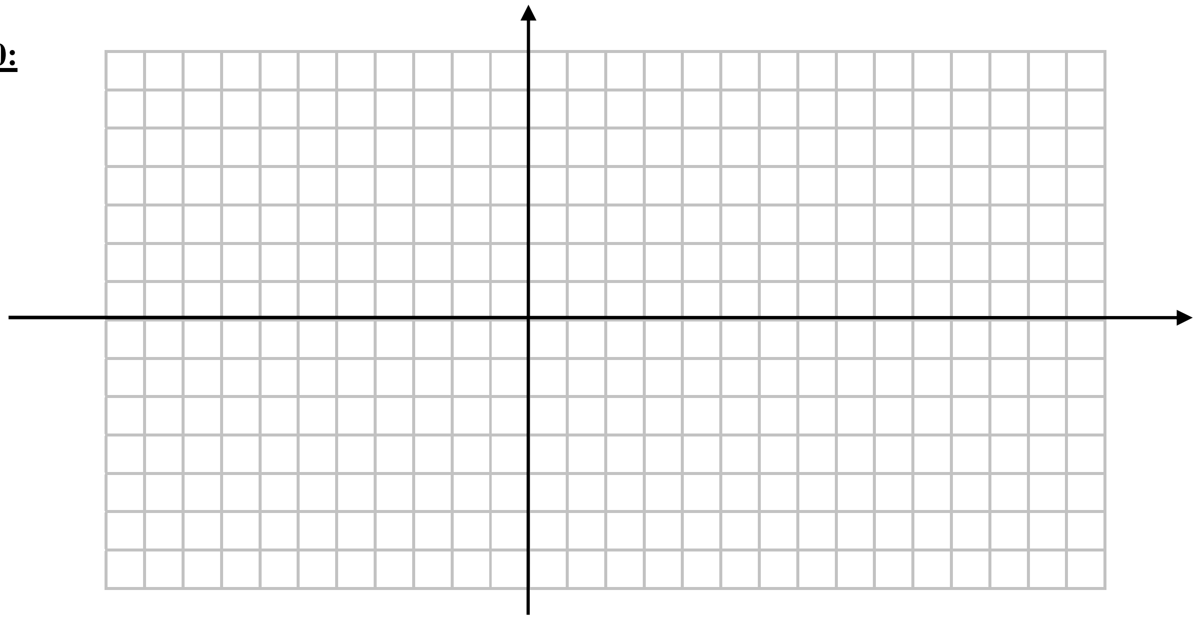
(but not continuous from the right at 2)

Continuity on an interval:

<u>Definition</u>: A function f is continuous on an interval if it is continuous at every point in the interval.

(If f is defined only at one side of an endpoint, then only continuity from the left or right is needed for it to be continuous at the endpoint.)

Example 10:



On what intervals is the above function continuous?

Theorem: If f and g are continuous at a and c is a constant, then $f+g$, $f-g$, fg, cf are also continuous at a. The quotient $\frac{f}{g}$ is also continuous at a if $g(a) \neq 0$.

Theorem:
Polynomials, rational functions, root functions, and trigonometric functions are continuous at every number in their domains.

Theorem:
If f is continuous at $b = \lim_{x \to a} g(x)$, then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

Theorem:
If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example 11: Where is $f(x) = \sin\left(\frac{1}{x}\right)$ continuous? At each discontinuity, classify the type of discontinuity and state the condition for continuity that is violated.

Discontinuous at 0

$\lim_{x \to 0} f(x)$ does not exist

Example 12: Where is $f(x) = \sqrt{\frac{x^2+1}{(x-1)^3}}$ continuous? At each discontinuity, classify the type of discontinuity and state the condition for continuity that is violated.

Domain: $(1, \infty)$

Note: $x^2+1 > 0$ for all x
(numerator always positive)

Denominator: need $x - 1 > 0 \Rightarrow x > 1$

This function is continuous on its domain.

Example 13: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \frac{5x^3 - 8x^2}{x - 7}$$

Domain: $\{x \mid x \neq 7\}$
$(-\infty, 7) \cup (7, \infty)$

Function is discontinuous at 7.
(infinite discontinuity)

$f(7)$ is not defined

$\lim_{x \to 7} f(x)$ does not exist

Example 14: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x)=\frac{x^3-3x^2}{x^2+4}$$

Domain: $(-\infty, \infty)$

(because $x^2+4>0$ for all x)

continuous everywhere

Example 15: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated. At which of these numbers is it continuous from the left or right?

$$f(x)=\begin{cases} x+3 & \text{if } x>0 \\ 4 & \text{if } x=0 \\ x^2+3 & \text{if } x<0 \end{cases}.$$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2+3) = 0^2+3 = 3$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+3) = 0+3 = 3$

So $\lim_{x\to 0} f(x) = 3$

However, $f(0)=4$. So $f(0) \neq \lim_{x\to 0} f(x)$

So f is discontinuous at 0.

removable discontinuity

Example 16: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated. At which of these numbers is it continuous from the left or right?

$$f(x)=\begin{cases} 5x & \text{if } x>1 \\ 5 & \text{if } x=1 \\ x+5 & \text{if } x<1 \end{cases}.$$

Example 17: Determine the values of x, if any, at which the function is discontinuous.

$$g(x)=\begin{cases}1 & x \text{ rational}\\ 0 & x \text{ irrational}\end{cases}$$

Discontinuous everywhere

Domain: $(-\infty, \infty)$

Example 18: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x)=\begin{cases}\dfrac{|x|}{x} & x \neq 0\\ 3 & x=0\end{cases}$$

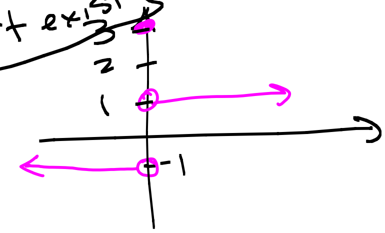
f is discontinuous at 0

jump discontinuity

$\lim\limits_{x \to 0} f(x)$ does not exist (because left-hand & right-hand limits don't exist)

Recall: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0\end{cases}$

$$f(x)=\begin{cases}\dfrac{|x|}{x}=\dfrac{-x}{x}=-1 & x<0\\ 3 & x=0\\ \dfrac{|x|}{x}=\dfrac{x}{x}=1 & x>0\end{cases}$$



Example 19: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x)=\begin{cases}\dfrac{x^2-25}{x-5} & x \neq 5\\ 10 & x=0\end{cases}$$

Example 20: Find a function g that agrees with f for $x \neq 1$ and is continuous on $(-\infty, \infty)$.

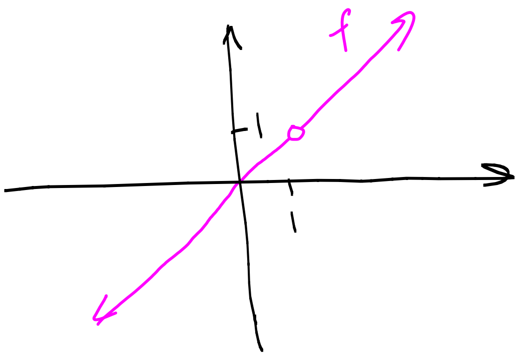
$$f(x) = \frac{x^2 - 1}{x - 1}.$$

Domain of f: $x \neq 1$

$(-\infty, 1) \cup (1, \infty)$

Let's graph f:

$f(x) = \frac{(x+1)(x-1)}{x-1}$



$f(1)$ is not defined, so f is not continuous at 1.

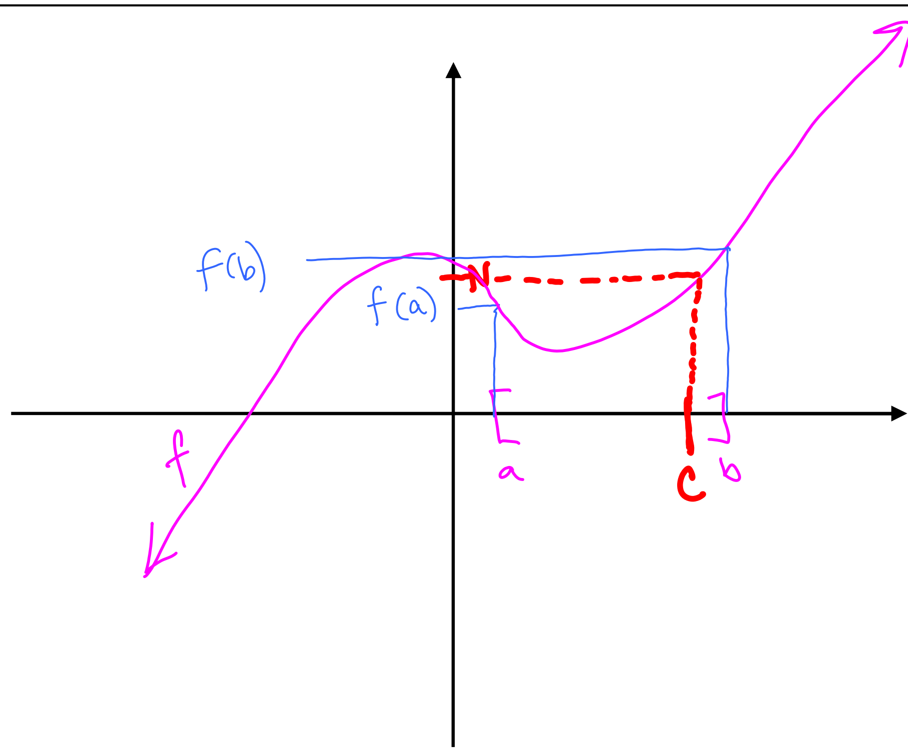
except at $x = 1$, $f(x)$ will be identical to $y = x + 1$

$\boxed{g(x) = x + 1}$

Domain of g: $(-\infty, \infty)$

The Intermediate Value Theorem:

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



Example 21: Show that $f(x) = x^3 - 4x^2 + 6$ has a zero between 1 and 2.

f is continuous on $(-\infty, \infty)$ and so certainly continuous on $[1, 2]$.

we want to show there is a c in $(1,2)$ such that $f(c) = 0$.

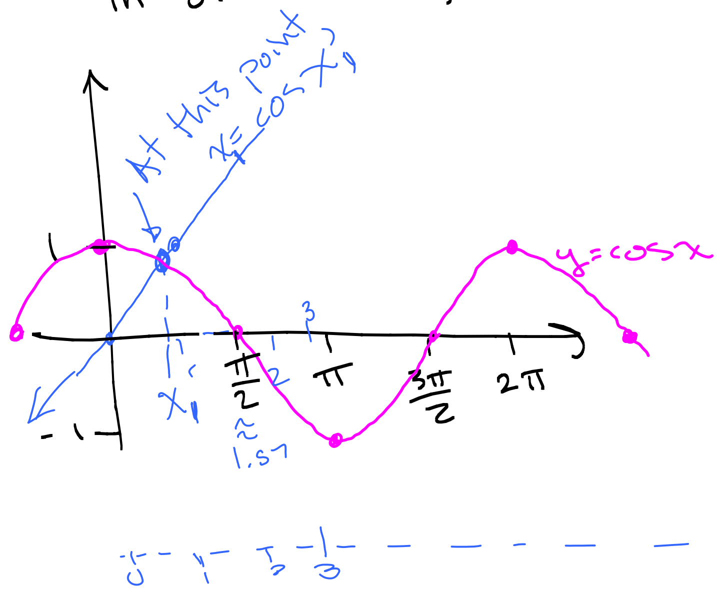
Need to find a and b so that for f(a) and f(b), 0 is in between them.

$f(1) = 1^3 - 4(1)^2 + 6 = 1 - 4 + 6 = 3$
$f(2) = 2^3 - 4(2)^2 + 6 = 8 - 16 + 6 = -2$

So, from Intermediate Value Theorem (IVT), there must be a c in $(1,2)$ such that $f(c) = 0$.

Example 22: Is there a number that is equal to its own cosine?

In other words, does the equation $x = \cos(x)$ have a solution?



Let $f(x) = x - \cos(x)$. Does $f(x)$ have a zero?

$f(0) = 0 - \cos(0) = 0 - 1 = -1$

$f(\pi) = \pi - \cos(\pi) = \pi - (-1) = \pi + 1$
≈ 4.1592653589

So, from IVT, there must be a c in $(0, \pi)$ such that $f(c) = 0$.
(such that $c = \cos(c)$)