1.5: Infinite Limits

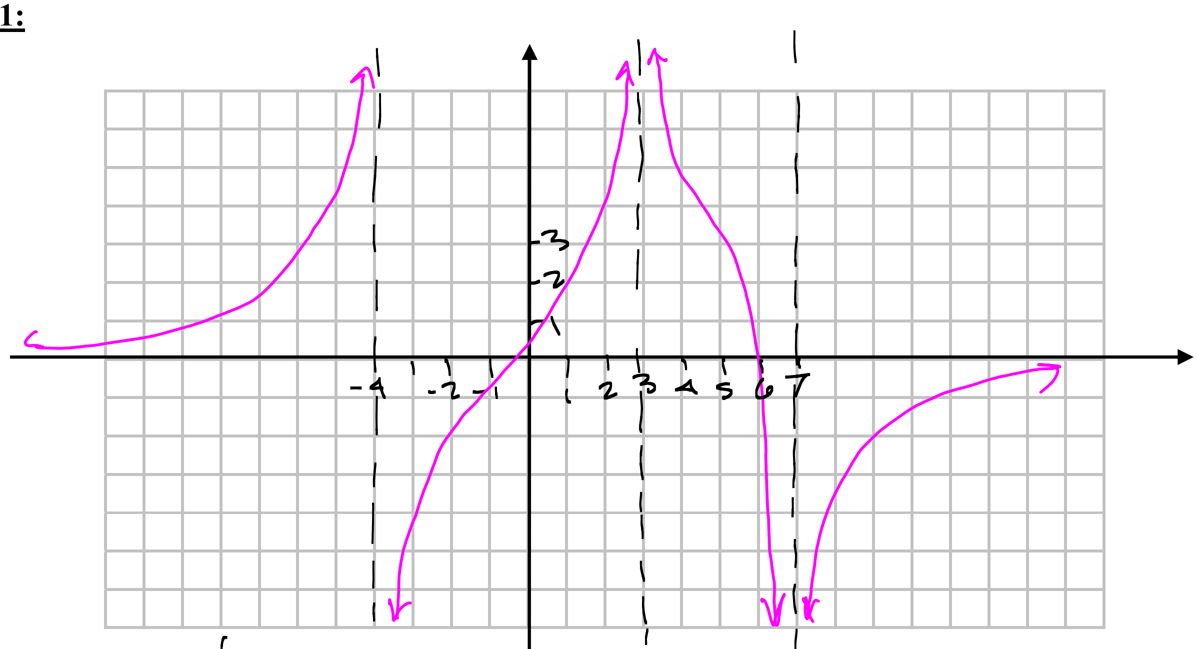
There are two types of limits involving infinity.

Limits at infinity, written in the form $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$, are related to horizontal asymptotes and will be covered in Section 3.5, as we learn to graph functions.

Infinite limits take the form of statements like $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. Infinite limits can result in vertical asymptotes, also important in graphing functions.

Determining infinite limits from a graph:

Example 1:



$\lim_{x\to -4} f(x)$ does not exist

$\lim_{x\to -4^-} f(x) = \infty$ (does not exist)

$\lim_{x\to -4^+} f(x) = -\infty$ (does not exist)

$\lim_{x\to 3} f(x) = \infty$

$\lim_{x\to 3^-} f(x) = \infty$

$\lim_{x\to 3^+} f(x) = \infty$

don't exist

$\lim_{x\to 7} f(x) = -\infty$

$\lim_{x\to 7^-} f(x) = -\infty$

$\lim_{x\to 7^+} f(x) = -\infty$

none of these limits exist

Determining infinite limits from a table of values:

Example 2: Use a table of values to determine $\lim\limits_{x\to 2} f(x)=\frac{x+6}{x-2}$. $\lim\limits_{x\to 2}\left(\frac{x+6}{x-2}\right)$

x	$f(x)=\frac{x+6}{x-2}$
1.9	
1.95	
1.99	↓
1.999	
1.9999	$-\infty$
⋮	
⋮	$+\infty$
2.000	↑
2.001	
2.01	
2.05	
2.10	

$\lim\limits_{x\to 2^-}\left(\frac{x+6}{x-2}\right) = -\infty$
$\lim\limits_{x\to 2^+}\left(\frac{x+6}{x-2}\right) = +\infty$ } these limits don't exist

So $\lim\limits_{x\to 2}\left(\frac{x+6}{x-2}\right)$ does not exist

Example 3: Use a table of values to determine $\lim\limits_{x\to 2} f(x)=\frac{x+6}{(x-2)^2}$. $\lim\limits_{x\to 2}\left(\frac{x+6}{(x-2)^2}\right)$

From table, $\lim\limits_{x\to 2^-}\left(\frac{x+6}{(x-2)^2}\right) = +\infty$
$\lim\limits_{x\to 2^+}\left(\frac{x+6}{(x-2)^2}\right) = +\infty$ } (none of these limits exist.)

So $\lim\limits_{x\to 2}\left(\frac{x+6}{(x-2)^2}\right) = +\infty$

Important:

Statements such as $\lim\limits_{x\to a} f(x)=\infty$, $\lim\limits_{x\to a} f(x)=-\infty$, or $\lim\limits_{x\to a^+} f(x)=-\infty$ do NOT mean the limit exists. Rather, these statements mean that the limit DOES NOT EXIST. and they describe the reason that the limit fails to exist (by describing the behavior of the function near the given *x*-value).

Formal definition of an infinite limit:

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive real number M, there exists a number $\delta > 0$ such that

$$f(x) > M \text{ whenever } 0 < |x - a| < \delta .$$

Similarly, $\lim_{x \to a} f(x) = \overset{-\infty}{\cancel{\infty}}$ means that for every negative real number N, there exists a number $\delta > 0$ such that

$$f(x) < N \text{ whenever } 0 < |x - a| < \delta .$$

<u>**Example 4:**</u> Prove that $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

If direct substitution results in $\frac{c}{0}$ with $c \neq 0$, the limit does not exist. You need to decide if you can put ∞ or $-\infty$ after the equals sign. Direct substitution

Evaluating infinite limits from an equation:

Example 5: Determine $\lim\limits_{x\to 4} \dfrac{x-8}{x-4}$.

$\dfrac{4-8}{4-4} \Rightarrow \dfrac{-4}{0}$ limit does not exist.

As $x \to 4^-$:

Try $x = 3.9$: $\dfrac{3.9-8}{3.9-4} \to \dfrac{-4}{-\text{tiny}} \to +\text{huge}$

So $\boxed{\lim\limits_{x\to 4^-}\left(\dfrac{x-8}{x-4}\right) = +\infty}$

As $x \to 4^+$:

Try 4.01

$x = 4.01 \Rightarrow \dfrac{4.01-8}{4.01-4} \to \dfrac{-4}{+\text{tiny}} \Rightarrow -\text{huge}$

So $\boxed{\lim\limits_{x\to 4^+}\left(\dfrac{x-8}{x-4}\right) = -\infty}$

So $\lim\limits_{x\to 4}\left(\dfrac{x-8}{x-4}\right)$ does not exist

Example 6: Determine $\lim\limits_{x\to 2^-} \dfrac{x-8}{x^2-4}$.

As $x \to 2^-$:

$x = 1.99 \longrightarrow \dfrac{1.99-8}{(1.99)^2-4} \to \dfrac{-6}{-\text{tiny}} \to +\infty$

$$\lim_{x\to 2^-} \frac{x-8}{x^2-4} = +\infty$$

Example 7: Determine $\lim\limits_{x\to 3} \dfrac{x^3-2}{(x-3)^2}$.

As $x \to 3^-$:

$x = 2.99 \longrightarrow \dfrac{(2.99)^3-2}{(2.99-3)^2} \to \dfrac{27-2}{(-0.01)^2} \longrightarrow \dfrac{25}{(-\text{tiny})^2} \longrightarrow \dfrac{25}{+\text{tiny}} \to +\text{huge}$

So $\lim\limits_{x\to 3^-} \dfrac{x^3-2}{(x-3)^2} = +\infty$

As $x \to 3^+$:

$x = 3.01 \longrightarrow \dfrac{(3.01)^3-2}{(3.01-3)^2} \to \dfrac{27-2}{(+\text{tiny})^2} \longrightarrow \dfrac{25}{+\text{tiny}} \longrightarrow +\text{huge}$

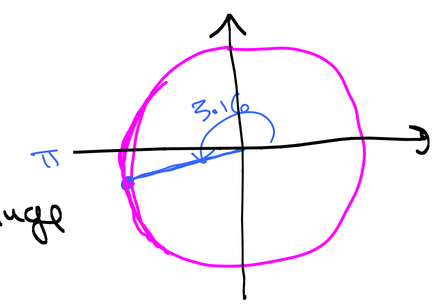
So $\lim\limits_{x\to 3^+} \dfrac{x^3-2}{(x-3)^2} = +\infty$

$\boxed{\lim\limits_{x\to 3} \dfrac{x^3-2}{(x-3)^2} = \infty}$ (limit does not exist)

Example 8: Determine $\lim\limits_{x\to \pi^+} \dfrac{\sin\left(\frac{x}{3}\right)}{1+\cos x}$.

$\lim\limits_{x\to \pi^+} \dfrac{\sin(x/3)}{1+\cos x} = \boxed{+\infty}$

Try: $x \to 3.16 \Rightarrow \dfrac{\sin\left(\frac{3.16}{3}\right)}{1+\cos(3.16)} \longrightarrow \dfrac{\sqrt{3}/2}{1+(-0.99)} \to \dfrac{\sqrt{3}/2}{+\text{tiny}} \to +\text{huge}$



Example 9: Determine $\lim\limits_{x\to \pi/6} \tan(3x)$.

See next page

Example 9: $\lim_{x \to \frac{\pi}{6}} \tan(3x)$

We will work this in class Thursday.

left side: $\lim_{x \to \frac{\pi}{6}^-} \left(\tan(3x)\right) = \lim_{x \to \frac{\pi}{6}^-} \left(\frac{\sin(3x)}{\cos(3x)}\right) = +\infty$

$x = \frac{\pi}{6} - \text{tiny} \Rightarrow \frac{\sin(\frac{\pi}{2} - \text{tiny})}{\cos(\frac{\pi}{2} - \text{tiny})} \longrightarrow \frac{1}{+\text{tiny}} \rightarrow + \text{huge}$

Direct sub

$x = \frac{\pi}{6} \Rightarrow \frac{\sin(\frac{3\pi}{6})}{\cos(3\pi/6)} \rightarrow \frac{\sin(\pi/2)}{\cos(\pi/2)} \rightarrow \frac{1}{0}$ (limit does not exist

right side: $\lim_{x \to \frac{\pi}{6}^+} \left(\tan(3x)\right) = \lim_{x \to \frac{\pi}{6}^+} \left(\frac{\sin(3x)}{\cos(3x)}\right) = -\infty$

$x = \frac{\pi}{6} + \text{tiny} \rightarrow \frac{\sin(\frac{\pi}{2} + \text{tiny})}{\cos(\frac{\pi}{2} + \text{tiny})} \longrightarrow \frac{1}{-\text{tiny}} \rightarrow - \text{huge}$

$\boxed{\text{so } \lim_{x \to \frac{\pi}{6}} \left(\tan(3x)\right) \text{ does not exist.}}$

Vertical asymptotes:

Vertical Asymptotes:

The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$\lim_{x\to a^-} f(x) = \infty$ $\quad\quad$ $\lim_{x\to a^+} f(x) = \infty$ $\quad\quad$ $\lim_{x\to a} f(x) = \infty$

$\lim_{x\to a^-} f(x) = -\infty$ $\quad\quad$ $\lim_{x\to a^+} f(x) = -\infty$ $\quad\quad$ $\lim_{x\to a} f(x) = -\infty$

Example 10: Determine the asymptotes of $f(x) = \dfrac{x-2}{x+3}$. Sketch the graph.

Vertical Asymptote: $x = -3$

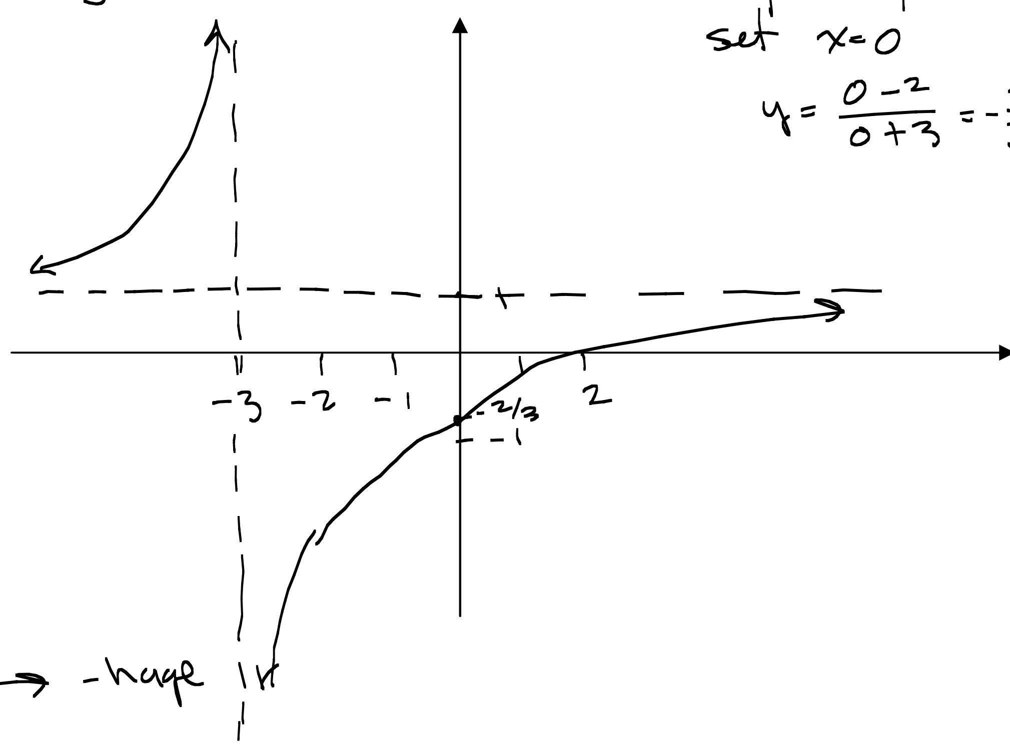
Find y-intercept: set $x=0$

$$y = \frac{0-2}{0+3} = -\frac{2}{3}$$

$$\lim_{x\to -3^-} f(x) = +\infty$$

$$x = -3.01 \Rightarrow \frac{-3.01-2}{-3.01+3}$$

$$\rightarrow \frac{-5}{-\text{tiny}} \Rightarrow +\text{huge}$$



$$\lim_{x\to -3^+} f(x) = -\infty$$

$$x = -2.99 \Rightarrow \frac{-2.99-2}{-2.99+3}$$

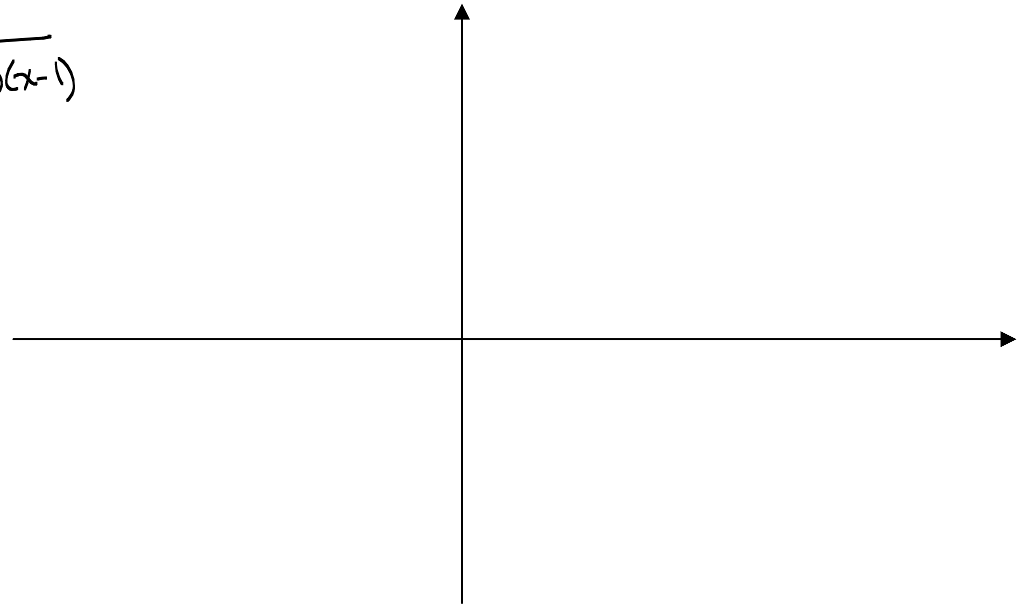
$$\Rightarrow \frac{-5}{+\text{tiny}} \rightarrow -\text{huge}$$

Example 11: Determine the vertical asymptotes of $f(x)=\frac{3}{x^2-1}$. Sketch the graph.

$f(x)=\frac{3}{x^2-1}=\frac{3}{(x+1)(x-1)}$

Vertical asymptotes:

$\boxed{\begin{array}{l} x=1 \\ x=-1 \end{array}}$



Example 12: Determine the vertical asymptotes of $f(x)=\frac{x^2-9}{x^2-5x+6}$. Sketch the graph.

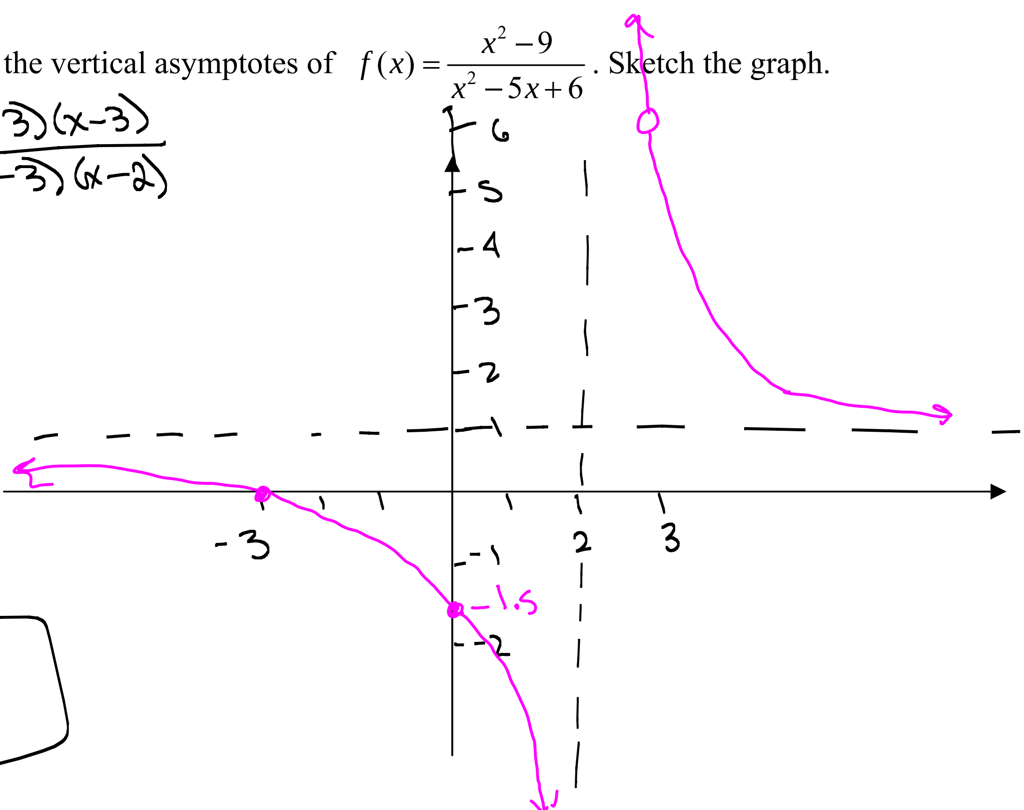
$f(x)=\frac{x^2-9}{x^2-5x+6}=\frac{(x+3)(x-3)}{(x-3)(x-2)}$

similar to

$g(x)=\frac{x+3}{x-2}$ except for domain

$\boxed{\text{Vertical asymptote: } x=2}$

$\boxed{\text{Removable discontinuity at } 3}$



Find y-intercept: set $x=0$:

$y=\frac{0+3}{0-2}=-\frac{3}{2}=-1.5$

Find x-intercept: set $y=0$: $0=\frac{x+3}{x-2}$

$0=x+3$

$-3=x$

where is removable discontinuity?

$g(3)=\frac{3+3}{3-2}$

$=\frac{6}{1}=6$

$(3,6)$