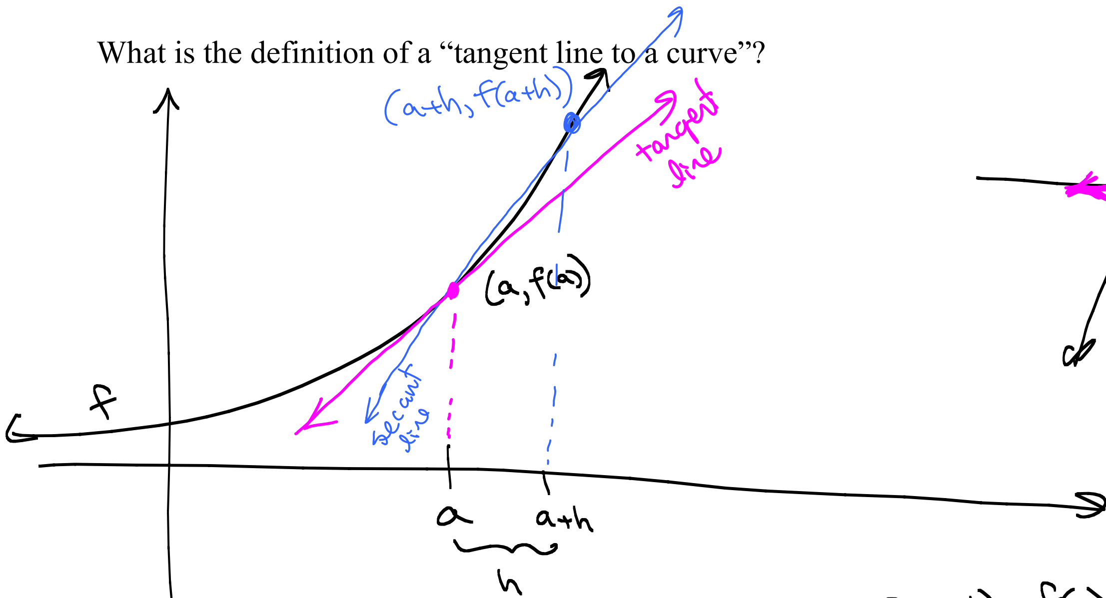
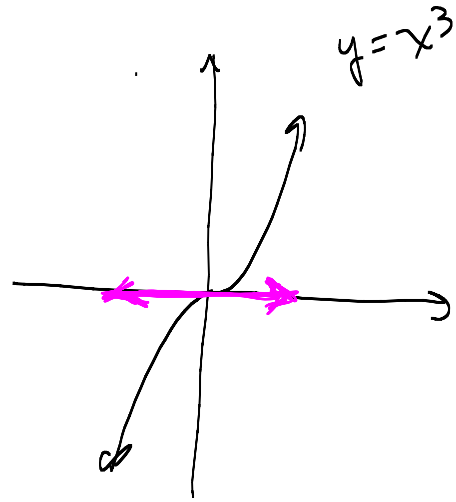
2.1: The Derivative and the Tangent Line Problem

What is the definition of a "tangent line to a curve"?





$$\text{Slope of secant line} = \frac{y_2-y_1}{x_2-x_1} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

To answer the difficulty in writing a clear definition of a tangent line, we can define it as the limiting position of the secant line as the second point approaches the first.

<u>Definition</u>: The tangent line to the curve $y = f(x)$ at the point $\underset{P}{(a, f(a))}$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ provided this limit exists.

Equivalently,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ provided this limit exists.

<u>Note</u>: If the tangent line is vertical, this limit does not exist. In the case of a vertical tangent, the equation of the tangent line is $x = a$.

<u>Note</u>: The slope of the tangent line to the graph of *f* at the point $(a, f(a))$ is also called the <u>slope of the graph</u> of *f* at $x = a$.

<u>How to get the second expression for slope</u>: Instead of using the points $(a, f(a))$ and $(x, f(x))$ on the secant line and letting $x \to a$, we can use $(a, f(a))$ and $(a+h, f(a+h))$ and let $h \to 0$.

$(3+h)(3+h)$
$= 9 + 3h + 3h + h^2$
$= 9 + 6h + h^2$

Find the equation of the tangent line:

$x_1 = 3$, $y_1 = 37$, $m = 24$

$y - y_1 = m(x - x_1)$
$y - 37 = 24(x-3)$
$y - 37 = 24x - 72$
$\boxed{y = 24x - 35}$ equation of tangent line

Example 1: Find the slope of the curve $y = 4x^2 + 1$ at the point $(3,37)$. Find the equation of the tangent line at this point.

$$m = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\overbrace{4(3+h)^2+1}^{f(3+h)} - \overbrace{[4(3)^2+1]}^{f(3)}}{h}$$

$$= \lim_{h\to 0} \frac{4(9+6h+h^2)+1 - [36+1]}{h} = \lim_{h\to 0} \frac{36+24h+4h^2+1-37}{h}$$

$$= \lim_{h\to 0} \frac{24h+4h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(24+4h)}{\cancel{h}} = \lim_{h\to 0} (24+4h)$$

$$= 24 + 4(0) = 24$$

$\boxed{\text{Slope of the curve: } 24}$ (Really the slope of the tangent line)

Example 2: Find an equation of the tangent line to the curve $y = x^3$ at the point $(1,1)$.

$y = x^3$

Let's try the other version of the definition

Here, $a = 1$

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$$

Recall:
$a^3 - b^3 = (a-b)(a^2+ab+b^2)$
$a^3 + b^3 = (a+b)(a^2-ab+b^2)$

$$= \lim_{x\to 1} \frac{x^3 - 1}{x - 1}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}}$$

$$= \lim_{x\to 1} (x^2+x+1) = 1^2+1+1 = 3$$

Find eqn of line:
$y - y_1 = m(x - x_1)$
$y - 1 = 3(x-1)$
$y - 1 = 3x - 3$
$\boxed{y = 3x - 2}$ eqn of tangent line

Example 3: Determine the equation of the tangent line to $f(x) = \sqrt{x}$ at the point where $x = 2$.

The derivative:

The derivative of a function at x is the slope of the tangent line at the point $(x, f(x))$. It is also the instantaneous rate of change of the function at x.

> Definition: The *derivative* of a function f at x is the function $f\,'$ whose value at x is given by
>
> "f prime"
>
> $$f\,'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \text{ provided this limit exists.}$$

The process of finding derivatives is called differentiation. To differentiate a function means to find its derivative.

Equivalent ways of defining the derivative:

$$f\,'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$ (Our book uses this one. It is identical to the definition above, except uses Δx in place of h.)

$$f\,'(x) = \lim_{w\to x}\frac{f(w)-f(x)}{w-x}$$

$$f\,'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$ (Gives the derivative at the specific point where $x = a$.)

$$f\,'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ (Gives the derivative at the specific point where $x = a$.)

Example 4: Suppose that $g(x) = \dfrac{x^2-6x}{3}$. Determine $g\,'(x)$ and $g\,'(3)$.

Note: $g(x) = \frac{1}{3}(x^2-6x)$
$= \frac{1}{3}x^2 - 2x$

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\overbrace{\tfrac{1}{3}(x+h)^2-2(x+h)}^{g(x+h)} - \overbrace{\left[\tfrac{1}{3}x^2-2x\right]}^{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{\tfrac{1}{3}(x^2+2xh+h^2)-\cancel{2x}-2h-\tfrac{1}{3}x^2+\cancel{2x}}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{\tfrac{1}{3}x^2}+\tfrac{2}{3}xh+\tfrac{1}{3}h^2-2h-\cancel{\tfrac{1}{3}x^2}}{h} = \lim_{h\to 0}\frac{\tfrac{2}{3}xh+\tfrac{1}{3}h^2-2h}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}\left(\tfrac{2}{3}x+\tfrac{1}{3}h-2\right)}{\cancel{h}} = \lim_{h\to 0}\left(\tfrac{2}{3}x+\tfrac{1}{3}h-2\right) = \tfrac{2}{3}x+\tfrac{1}{3}(0)-2 = \tfrac{2}{3}x-2$$

$$\boxed{g'(x) = \tfrac{2}{3}x-2}$$

Now find $g'(3)$:

$g'(x) = \frac{2}{3}x - 2$

$g'(3) = \frac{2}{3}(3) - 2$

$= 2-2$

$= \boxed{0}$

Note: $f(2) = \sqrt{2^2+1} = \sqrt{5}$

So our point is $(2, \sqrt{5})$

Example 5: Suppose that $f(x) = \sqrt{x^2+1}$. Find the equation of the tangent line at the point where $x = 2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} \left(\frac{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \right) = \lim_{h \to 0} \frac{(x+h)^2+1 - (x^2+1)}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{+1} \cancel{-x^2} \cancel{-1}}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)} = \lim_{h \to 0} \frac{2xh + h^2}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)} = \lim_{h \to 0} \frac{2x+h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} = \frac{2x+0}{\sqrt{(x+0)^2+1} + \sqrt{x^2+1}} = \frac{2x}{2\sqrt{x^2+1}}$$

$$\boxed{f'(x) = \frac{x}{\sqrt{x^2+1}}}$$

To find slope where $x=2$, substitute 2:

$$m = f'(2) = \frac{2}{\sqrt{2^2+1}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

From earlier, $(x_1, y_1) = (2, \sqrt{5})$

Find eqn of tangent line

$$y - y_1 = m(x - x_1)$$

$$y - \sqrt{5} = \frac{2\sqrt{5}}{5}(x-2)$$

$$y - \sqrt{5} = \frac{2\sqrt{5}}{5}x - \frac{4\sqrt{5}}{5}$$

$$y = \frac{2\sqrt{5}}{5}x - \frac{4\sqrt{5}}{5} + \frac{\sqrt{5}}{1}\left(\frac{5}{5}\right)$$

$$\boxed{y = \frac{2\sqrt{5}}{5}x + \frac{\sqrt{5}}{5}}$$

Example 6: Determine the equation of the tangent line to $f(x) = \dfrac{x-2}{x^2+1}$ at the point $\left(-2, -\dfrac{4}{5}\right)$.

Summary:

The slope of the secant line between two points is often called a <u>difference quotient</u>. The <u>difference quotient of *f* at *a*</u> can be written in either of the forms below.

$$\frac{f(x)-f(a)}{x-a} \qquad\qquad \frac{f(a+h)-f(a)}{h}.$$

Both of these give the slope of the secant line between two points: $(x, f(x))$ and $(a, f(a))$ or, alternatively, $(a, f(a))$ and $(a+h, f(a+h))$.
The slope of the secant line is also the average rate of change of *f* between the two points.

The <u>derivative of *f* at *a*</u> is:

1) the limit of the slopes of the secant lines as the second point approaches the point $(a, f(a))$.

2) the slope of the tangent line to the curve $y = f(x)$ at the point where $x = a$. (often called the slope of the curve)

3) the (instantaneous) rate of change of f with respect to *x* at *a*.

4) $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ (limit of the difference quotient)

5) $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ (limit of the difference quotient)

Common notations for the derivative of $y = f(x)$:

$f'(x) \qquad \frac{d}{dx}f(x) \qquad y' \qquad D_x f(x) \qquad \frac{dy}{dx} \qquad Df(x)$

The notation $\frac{dy}{dx}$ was created by Gottfried Wilhelm Leibniz and means $\frac{dy}{dx} = \cancel{\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta y}}$. $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$

To evaluate the derivative at a particular number *a*, we write

Ex: Suppose $f'(x) = x^3 - 3x$

$$f'(a) \text{ or } \left.\frac{dy}{dx}\right|_{x=a}$$

$f'(4) = \left.(x^3-3x)\right|_{x=4} = 4^3 - 3(4) = 64 - 12 = \boxed{52}$

Differentiability:

<u>Definition</u>: A function *f* is *differentiable* at *a* if $f'(a)$ exists. It is *differentiable on an open interval* if it is differentiable at every number in the interval.

<u>Theorem</u>: If *f* is differentiable at *a*, then *f* is continuous at *a*.

<u>Note</u>: The converse is not true—there are functions that are continuous at a number but not differentiable.

<u>Note</u>: Open intervals: (a,b), $(-\infty,a)$, (a,∞), $(-\infty,\infty)$.

Closed intervals: $[a,b]$, $(-\infty,a]$, $[a,\infty)$, $(-\infty,\infty)$.

To discuss differentiability on a closed interval, we need the concept of a *one-sided derivative*.

Derivative from the left: $\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$

Derivative from the right: $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$

For a function *f* to be differentiable on the closed interval $[a,b]$, it must be differentiable on the open interval (a,b). In addition, the derivative from the right at *a* must exist, and the derivative from the left at *b* must exist.

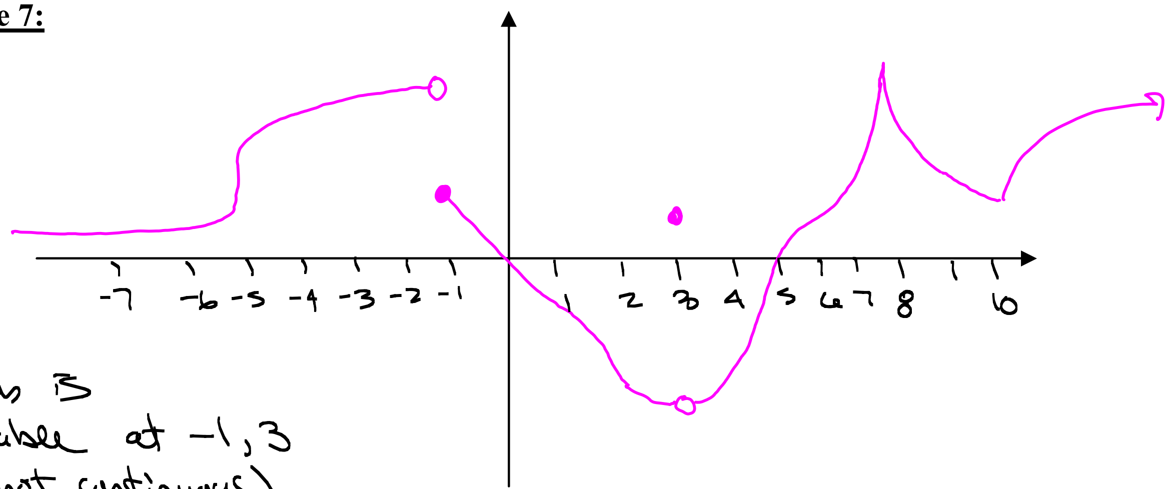
<u>Ways in which a function can fail to be differentiable</u>:

1. Sharp corner
2. Cusp
3. Vertical tangent (vertical lines have undefined slope)
4. Discontinuity

Cusp: the derivative at a is undefined, because the slopes of the tangent line as $x \to a^-$ approach $+\infty$, while the slopes as $x \to a^+$ approach $-\infty$

(or vice versa)

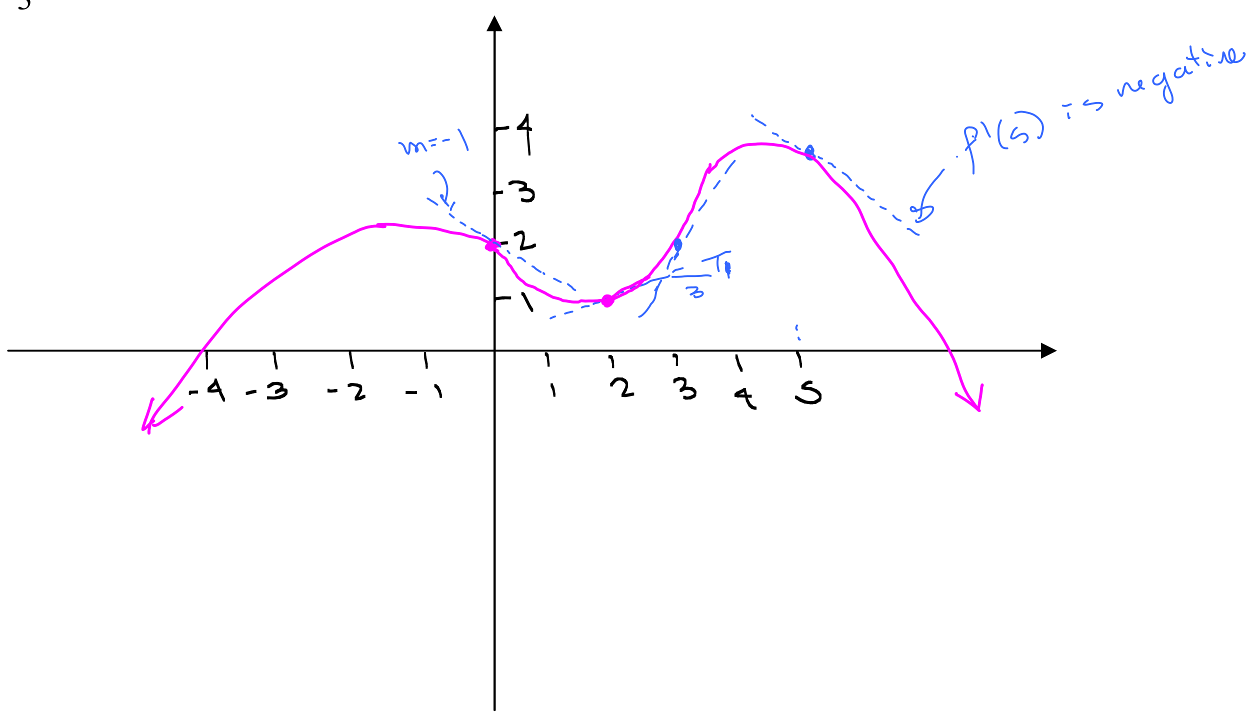
Example 7:



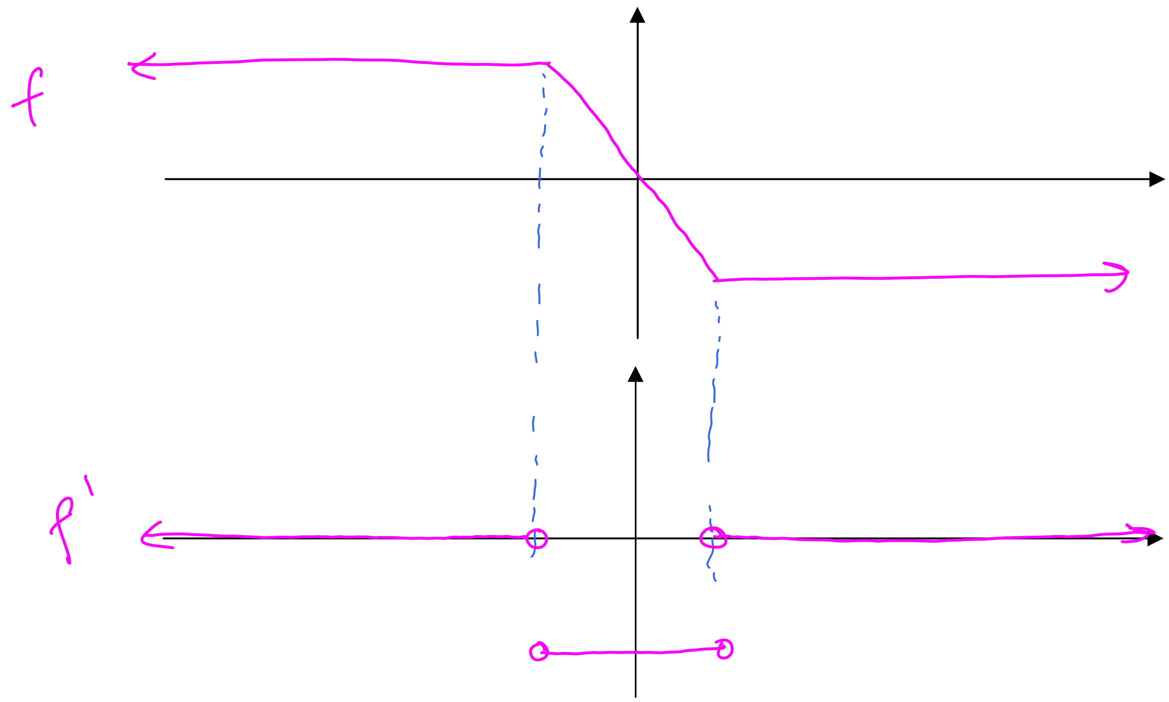
This function is not differentiable at −1, 3 (not continuous)

Also not differentiable at
8 (cusp)
10 (sharp corner)
−5 (vertical tangent)

Example 8: Sketch the graph of a function for which $f(0)=2$, $f'(0)=-1$, $f(2)=1$, $f'(2)=\frac{1}{3}$, $f'(3)>f'(2)$, and $f'(5)<0$.



Example 9: Use the graph of the function to draw the graph of the derivative.



Example 10: Use the graph of the function to draw the graph of the derivative.