2.2: Basic Differentiation Rules and Rates of Change

Basic differentiation formulas:

1. $\frac{d}{dx}(c) = 0$ for any constant c.

2. $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real number n. (Power rule)

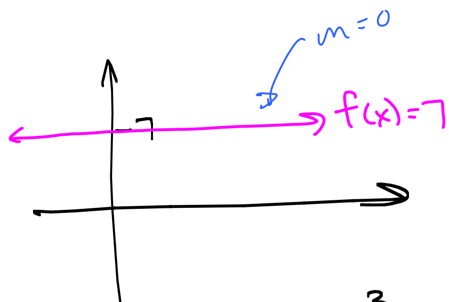
3. $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$

4. $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

5. $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Example 1: Find the derivative of $f(x) = 7$.

$$f'(x) = 0$$



Ex $1\frac{1}{2}$: $f(x) = x^5$

$$f'(x) = 5x^{5-1} = \boxed{5x^4}$$

Ex $1\frac{3}{4}$: $g(x) = 7x^3$

$$g'(x) = 7\frac{d}{dx}(x^3) = 7(3x^2) = \boxed{21x^2}$$

Example 2: Find the derivative of $f(x) = 5x^3 - x^7 + 12x$.

$$f'(x) = 15x^2 - 7x^6 + 12x^0 = \boxed{15x^2 - 7x^6 + 12}$$

Note: $x^0 = 1$ for $x \neq 0$

Note: If $g(x) = 12x$ then $g'(x) = 12$.

Example 3: Find the derivative of $g(x) = x^{17} + x^{3/2}$.

$$g(x) = x^{17} + x^{3/2}$$

$$g'(x) = 17x^{16} + \frac{3}{2}x^{\frac{3}{2}-1} = 17x^{16} + \frac{3}{2}x^{\frac{3}{2}-\frac{2}{2}} = \boxed{17x^{16} + \frac{3}{2}x^{\frac{1}{2}}} = \boxed{17x^{16} + \frac{3\sqrt{x}}{2}}$$

Recall:

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$

$$\frac{1}{x^n} = x^{-n}$$

Example 4: Find the derivative of $f(x) = \sqrt[5]{x} + \dfrac{1}{x^2}$.

$f(x) = \sqrt[5]{x} + \dfrac{1}{x^2}$

Rewrite: $f(x) = x^{\frac{1}{5}} + x^{-2}$

$f'(x) = \frac{1}{5}x^{\frac{1}{5}-1} - 2x^{-2-1}$

$= \frac{1}{5}x^{-4/5} - 2x^{-3}$

$= \dfrac{1}{5x^{4/5}} - \dfrac{2}{x^3}$

$= \boxed{\dfrac{1}{5\sqrt[5]{x^4}} - \dfrac{2}{x^3}}$

Example 5: Find the derivative of $f(x) = \dfrac{2}{\sqrt[4]{x}}$.

$f(x) = \dfrac{2}{\sqrt[4]{x}} = 2x^{-\frac{1}{4}}$

$f'(x) = 2\left(-\frac{1}{4}\right)x^{-\frac{1}{4}-1} = \boxed{-\frac{1}{2}x^{-\frac{5}{4}}} = -\dfrac{1}{2\sqrt[4]{x^5}}$

Example 6: Find the derivative of $h(x) = \left(\sqrt{x}\right)^5$.

$h(x) = (x^{\frac{1}{2}})^5 = x^{5/2}$

$h'(x) = \frac{5}{2}x^{\frac{5}{2}-1} = \boxed{\frac{5}{2}x^{3/2}} = \dfrac{5(x^3)^{1/2}}{2} = \boxed{\dfrac{5\sqrt{x^3}}{2}}$

Example 7: Find the derivative of $f(x) = -\sqrt[3]{6x^4}$.

$f(x) = -\sqrt[3]{6}\sqrt[3]{x^4}$

$= -\sqrt[3]{6}\, x^{4/3}$

$f'(x) = -\sqrt[3]{6}\left(\frac{4}{3}x^{\frac{4}{3}-1}\right) = -\sqrt[3]{6}\left(\frac{4}{3}x^{1/3}\right)$

$= -\frac{4}{3}\sqrt[3]{6}\sqrt[3]{x}$

$= -\frac{4}{3}\sqrt[3]{6x} = \boxed{-\dfrac{4\sqrt[3]{6x}}{3}}$

Note:

$x^{\frac{a}{b}} = (x^a)^{1/b} = \sqrt[b]{x^a}$

or $x^{\frac{a}{b}} = (x^{\frac{1}{b}})^a = (\sqrt[b]{x})^a$

provided all are defined

Example 8: Find the derivative of $f(x) = \dfrac{10}{x^4}$.

$f(x) = 10x^{-4}$

$f'(x) = -40x^{-4-1}$

$= -40x^{-5} = \boxed{-\dfrac{40}{x^5}}$

$= -40\left(\dfrac{1}{x^5}\right) =$

Example 9: Find the derivative of $g(x) = \frac{2\sqrt{x}}{7}$.

$g(x) = \frac{2}{7}x^{1/2}$

$g'(x) = \frac{\cancel{2}}{7} \cdot \frac{1}{\cancel{2}} x^{-1/2} = \frac{1}{7} x^{-1/2} = \boxed{\frac{1}{7\sqrt{x}}}$

Example 10: Find the derivative of $f(t) = \frac{3}{4t^2} - \sqrt[3]{7t}$.

$f(t) = \frac{3}{4} t^{-2} - \sqrt[3]{7}\sqrt[3]{t}$

$= \frac{3}{4} t^{-2} - \sqrt[3]{7}\, t^{1/3}$

$f'(t) = \frac{3}{4} \cdot (-2t^{-3}) - \sqrt[3]{7}\left(\frac{1}{3} t^{-2/3}\right) = \boxed{-\frac{3}{2t^3} - \frac{\sqrt[3]{7}}{3\sqrt[3]{t^2}}}$

Example 11: Find the derivative of $f(u) = \frac{7u^5 + u^2 - 9\sqrt{u}}{u^2}$.

$f(u) = \frac{7u^5}{u^2} + \frac{u^2}{u^2} - \frac{9u^{1/2}}{u^2}$

$= 7u^3 + 1 - 9u^{\frac{1}{2}-2}$

$= 7u^3 + 1 - 9u^{-\frac{3}{2}}$

$f'(u) = 21u^2 + 0 - 9\left(-\frac{3}{2}\right)u^{-\frac{5}{2}} = 21u^2 + \frac{27}{2u^{5/2}} = \boxed{21u^2 + \frac{27}{2\sqrt{u^5}}}$

Example 12: Find the equation of the tangent line to the graph of $f(x) = 3x - x^2$ at the point $(-2, -10)$.

$f'(x) = 3 - 2x$

$m = f'(-2) = 3 - 2(-2) = 3 + 4 = 7$

$y - y_1 = m(x - x_1)$

$y - (-10) = 7(x - (-2))$

$y + 10 = 7(x + 2)$

$y + 10 = 7x + 14$

$\boxed{y = 7x + 4}$ eqn of tangent line

Example 13: Find the point(s) on the graph of $f(x) = x^2 + 6x$ where the tangent line is horizontal.

Slope of horizontal line is 0, so we want $f'(x) = 0$.

$$f'(x) = 2x + 6$$

Set $f'(x) = 0$: $2x + 6 = 0$

$$2x = -6$$

$$x = -3$$

Need the ordered pairs, so
Find the y-value: $f(-3) = (-3)^2 + 6(-3) = 9 - 18 = -9$.

$\boxed{\text{Tangent line is horizontal at the point } (-3, -9).}$

Definition: The *normal line* to a curve at the point *P* is defined to be the line passing through *P* that is perpendicular to the tangent line at that point.

Example 14: Determine the equation of the normal line to the curve $y = \dfrac{1}{x}$ at the point $\left(3, \dfrac{1}{3}\right)$.

$$y = \frac{1}{x} = x^{-1}$$

$$\frac{dy}{dx} = -1x^{-2} = -\frac{1}{x^2}$$

Slope of tangent line: $\left.\dfrac{dy}{dx}\right|_{x=3} = \left.\left(-\dfrac{1}{x^2}\right)\right|_{x=3} = -\dfrac{1}{3^2} = -\dfrac{1}{9}$

Slope of normal line: 9.

Recall:
(If two lines are perpendicular, their slopes are opposite reciprocals)

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{3} = 9(x - 3)$$

$$y - \frac{1}{3} = 9x - 27$$

$$y = 9x - \frac{81}{3} + \frac{1}{3}$$

$$\boxed{y = 9x - \frac{80}{3}}$$

Derivatives of trigonometric functions:

Memorize these

$\dfrac{d}{dx}(\sin x) = \cos x$	$\dfrac{d}{dx}(\csc x) = -\csc x \cot x$
$\dfrac{d}{dx}(\cos x) = -\sin x$	$\dfrac{d}{dx}(\sec x) = \sec x \tan x$
$\dfrac{d}{dx}(\tan x) = \sec^2 x$	$\dfrac{d}{dx}(\cot x) = -\csc^2 x$

Note: Derivatives of All the cofunctions have a minus sign.

Example 15: Find the derivative of $y = 2\cos x - 4\tan x$.

$$\frac{dy}{dx} = 2(-\sin x) - 4(\sec^2 x) = \boxed{-2\sin x - 4\sec^2 x}$$

Example 16: Find the derivative of $y = \frac{\sin x}{4} + 3x^4 + \pi^2$.

$$y = \frac{1}{4}\sin x + 3x^4 + \pi^2$$

$$\frac{dy}{dx} = \frac{1}{4}\cos x + 12x^3 + 0$$

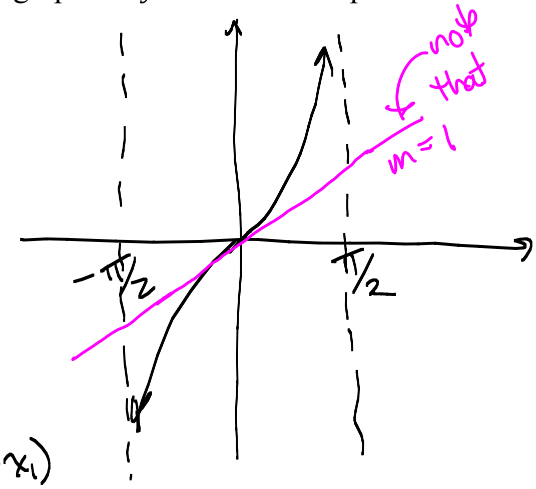
$$= \boxed{\frac{1}{4}\cos x + 12x^3}$$

Example 17: Determine the equation of the tangent line to the graph of ~~$y = \sec x$~~ ~~$y = \cot x$~~ $y = \tan x$ at the point where ~~$x = \frac{\pi}{4}$~~ $x = 0$

$$\frac{dy}{dx} = \sec^2 x$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \sec^2(0) = \left(\frac{1}{\cos(0)}\right)^2 = \left(\frac{1}{1}\right)^2 = 1$$

So $m = 1$



Find y-value for $x = 0$:

$$y = \tan(0) = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 1(x - 0)$$

$\boxed{y = x}$ eqn of tangent line

Example 18: Find the points on the curve $y = \tan x - 2x$ where the tangent line is horizontal.

Note: solve $t^2 = 7$

$$\sqrt{t^2} = \pm\sqrt{7}$$

$$t = \pm\sqrt{7}$$

$$\frac{dy}{dx} = \sec^2(x) - 2$$

Set $\frac{dy}{dx} = 0$: $0 = \sec^2(x) - 2$

$$2 = \sec^2(x)$$

$$2 = \left(\frac{1}{\cos(x)}\right)^2$$

$$2 = \frac{1}{\cos^2(x)}$$

Take reciprocal of both sides, or:

$$2\cos^2(x) = 1$$

$$\cos^2(x) = \frac{1}{2}$$

$$(\cos(x))^2 = \frac{1}{2}$$

$$\rightarrow \cos(x) = \pm\sqrt{\frac{1}{2}}$$

$$\cos(x) = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

multiples of $\frac{\pi}{4}$ ($\frac{1}{2}$ way thru quadrant)

$$\boxed{x = \frac{\pi}{4} + k\left(\frac{\pi}{2}\right) \quad k \text{ any integer} \quad k \in \mathbb{Z}}$$

($\mathbb{Z}$ = set of integers)

The derivative as a rate of change:

The <u>average rate of change</u> of $y = f(x)$ with respect to x over the interval $[x_0, x_1]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}, \text{ where } h = x_1 - x_0 \neq 0.$$

This is the same as the slope of the secant line joining points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$.

The <u>instantaneous rate of change</u> (or, equivalently, just the <u>rate of change</u>) of f when $x = a$ is the slope of the tangent line to graph of f at the point $(a, f(a))$.

Therefore, the instantaneous rate of change is given by the <u>derivative</u> f'.

Recall: Volume of sphere: $V = \frac{4}{3}\pi r^3$

<u>Example 19:</u> Find the average rate of change in volume of a sphere with respect to its radius r as r changes from 3 to 4. Find the instantaneous rate of change when the radius is 3.

Average rate of change (V = volume, r = radius) $= \frac{\Delta V}{\Delta r} = \frac{V_2 - V_1}{r_2 - r_1} = \frac{V(4) - V(3)}{4-3}$

$= \frac{\frac{4}{3}\pi(4)^3 - \frac{4}{3}\pi(3)^3}{1} = \frac{4}{3}(64)\pi - \frac{4}{3}(27)\pi$

$= \frac{4}{3}\pi(64 - 27) = \frac{4}{3}\pi(37)$

$= \boxed{\frac{148}{3}\pi}$

Instantaneous rate of change:

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = \frac{4}{3}\pi \frac{d}{dr}(r^3)$

$= \frac{4}{3}\pi(3r^2)$

$= 4\pi r^2$

$\left.\frac{dV}{dr}\right|_{r=3} = 4\pi r^2\Big|_{r=3}$

$= 4\pi(3)^2 = \boxed{36\pi}$

<u>Example 20:</u> Find the rate of change of the area of a circle with respect to (a) the diameter; (b) the circumference.

Area $= A = \pi r^2$

(a) Put this in terms of diameter:

radius $= \frac{1}{2}$(diameter) $\Rightarrow$ diameter $= 2$(radius)

$r = \frac{1}{2}d \Rightarrow A = \pi\left(\frac{1}{2}d\right)^2$ [using $A = \pi r^2$]

$A(d) = \pi\left(\frac{d^2}{4}\right) = \frac{\pi d^2}{4} = \frac{\pi}{4}(d^2)$

rate of change: $A'(d) = \frac{\pi}{4}(2d) = \boxed{\frac{\pi d}{2}}$

(b) Put $A = \pi r^2$ in terms of circumference C:

$C = 2\pi r$

$\frac{C}{2\pi} = r$

Substitute this into $A = \pi r^2$:

$A(C) = \pi\left(\frac{C}{2\pi}\right)^2$

$= \pi\left(\frac{C^2}{4\pi^2}\right) = \frac{C^2}{4\pi}$

$= \frac{1}{4\pi}(C^2)$

$A'(C) = \frac{1}{4\pi}(2C) = \frac{2C}{4\pi} = \boxed{\frac{C}{2\pi}}$

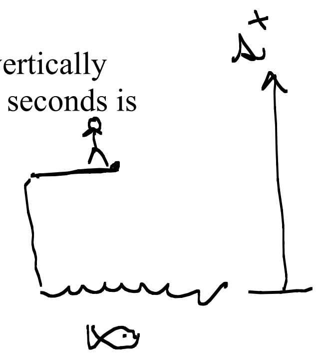
Velocity:

If the independent variable represents *time*, then the derivative can be used to analyze motion.

If the function $s(t)$ represents the position of an object, then the derivative $s'(t) = \frac{ds}{dt}$ is the velocity of the object.

(The velocity is the instantaneous rate of change in distance. The average velocity is the average rate of change in distance.)

with respect to time



Example 21: A person stands on a bridge 40 feet above a river. He throws a ball vertically upward with an initial velocity of 50 ft/sec. Its height (in feet) above the river after *t* seconds is $s = -16t^2 + 50t + 40$.

take derivative

a) What is the velocity after 3 seconds?
b) How high will it go?
c) How long will it take to reach a velocity of 20 ft/sec?
d) When will it hit the water? How fast will it be going when it gets there?

(a) $v(t) = \frac{ds}{dt} = -32t + 50$ (also equal to $s'(t)$)

$s'(t) = -32t + 50$)

$v(3) = \left.\frac{ds}{dt}\right|_{t=3} = s'(3) = -32(3)+50$
$= -96+50 = \boxed{-46 \text{ ft/sec}}$ so it is going down (because v is negative)

(b) At maximum height, velocity $= v(t) = s'(t) = 0$:

Set $s'(t) = 0$: $-32t + 50 = 0$

$50 = 32t$

$\frac{50}{32} = t$

$t = \frac{25}{16} = 1.5625$ time at max height

At $t = 1.5625$ seconds,
height is $s(1.5625) = -16(1.5625)^2 + 50(1.5625) + 40$
$= \boxed{79.0625 \text{ ft}}$ is maximum height

(c) Set $v(t) = s'(t) = 20$:

$-32t + 50 = 20$

$-32t = -30$

$t = \frac{-30}{-32} = \frac{15}{16}$

Reaches 20 ft/sec after $\boxed{\frac{15}{16} \text{ seconds}}$

(d) when it hits the water, $s(t) = 0$

Set $s(t) = 0$:

$0 = -16t^2 + 50t + 40$

Quadratic Formula:

$t = \frac{-50 \pm \sqrt{(50)^2 - 4(-16)(40)}}{2(-16)}$

see next page

Ex 21 cont'd

$\boxed{t \approx 3.785 \text{ sec}}$

$t = -0.66 \text{ sec}$

To get the velocity plug $t = 3.785$ into

$v(t) = s'(t) = -32t + 50$

$v(3.785) = -32(3.785) + 50 = \boxed{-71.12 \frac{ft}{sec}}$

Example 22: Suppose a bullet is shot straight up at an initial velocity of 73 feet per second. If air resistance is neglected, its height from the ground (in feet) after t seconds is given by $h(t) = -16.1t^2 + 73t$.

a. The velocity after 2 seconds.
b. How high will the bullet go?
c. When will the bullet reach the ground?
d. How fast will it be traveling when it hits the ground?

Velocity is $h'(t) = -32.2t + 73$

(a) $h'(2) = -32.2(2) + 73 = \boxed{8.6 \text{ ft/sec}}$

(b) At max height, $h'(t) = 0$:

$-32.2t + 73 = 0$

$-32.2t = -73$

$t = \frac{-73}{-32.2} \text{ sec} \approx 2.267 \text{ sec}$

Max height $= h\left(\frac{73}{32.2}\right) = -16.1\left(\frac{73}{32.2}\right)^2 + 73\left(\frac{73}{32.2}\right) \approx \boxed{82.748 \text{ ft}}$

(c) set $h(t) = 0$:

$-16.1t^2 + 73t = 0$

$t(-16.1t + 73) = 0$

$t = 0, \quad -16.1t + 73 = 0$

$-16.1t = -73$

$t = \frac{-73}{-16.1} = \frac{73}{16.1} \approx 4.534 \text{ sec}$

It hits the ground at approximately 4.534 sec.

(d) $h'\left(\frac{73}{16.1}\right) = -32.2\left(\frac{73}{16.1}\right) + 73$

$= -73 \text{ ft/sec}$

Speed is 73 ft/sec when it hits ground

Example 23: Suppose the position of a particle is given by $f(t) = t^4 - 32t + 7$. What is the velocity after 3 seconds? When is the particle at rest?

$f'(t) = 4t^3 - 32$

after 3 seconds, velocity is $f'(3) = 4(3)^3 - 32 = \boxed{76 \text{ units/sec}}$

To find when particle is at rest, set $f'(t) = 0$:

$4t^3 - 32 = 0$

$4t^3 = 32$

$t^3 = 8$

$t = 2$ seconds.

Particle is at rest when t = 2 seconds.