2.4: The Chain Rule

The chain rule allows us to find the derivative of many more functions.

Example 1: Suppose $g(x) = (x^3 + 2)^2$. Find $g'(x)$.

One way: $g(x) = (x^3+2)(x^3+2)$

$= x^6 + 4x^3 + 4$

$g'(x) = \boxed{6x^5 + 12x^2}$

what if I had $h(x) = (x^3+2)^{50}$

The chain rule lets us differentiate composite functions.

> The Chain Rule:
>
> If f and g are both differentiable and $F(x) = f(g(x))$, then F is differentiable and
>
> $$F'(x) = f'(g(x))\, g'(x).$$
>
> Alternatively, if $y = f(u)$ and $u = f(x)$ are both differentiable functions, then
>
> $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

With the chain rule, we take the derivative of the "outer function" and multiply by the derivative of the "inner function".

Example 2: Find the derivative of $h(x) = (x^3 + 2)^{50}$.

$h(x) = (x^3+2)^{50} = y$

Let $u = x^3+2$ (inside function)

$y = u^{50}$ (outside function)

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 50u^{49}(3x^2)$

$= 50(x^3+2)^{49}(3x^2)$

$= \boxed{150x^2(x^3+2)^{49}}$

OR (better) $h(x) = (x^3+2)^{50}$

$h'(x) = 50(x^3+2)^{49}\frac{d}{dx}(x^3+2)$

$= 50(x^3+2)^{49}(3x^2)$

$= \boxed{150x^2(x^3+2)^{49}}$

Ex 2½:

$$f(x)=3(2x^4-x)^5$$

$$f'(x)=15(2x^4-x)^4\frac{d}{dx}(2x^4-x)$$

$$=\boxed{15(2x^4-x)^4(8x^3-1)}$$

Example 3: Find the derivative of $f(x)=\sqrt{2x^3-5x}$.

$$f(x)=(2x^3-5x)^{1/2}$$

$$f'(x)=\frac{1}{2}(2x^3-5x)^{-\frac{1}{2}}\frac{d}{dx}(2x^3-5x)$$

$$=\boxed{\frac{1}{2}(2x^3-5x)^{-1/2}(6x^2-5)}=\boxed{\frac{6x^2-5}{2\sqrt{2x^3-5x}}}$$

Example 4: Suppose that $y=\dfrac{2}{(3x^5-4)^3}$. Find $\dfrac{dy}{dx}$.

Example 5: Find the derivative of $f(x)=\sqrt[3]{\cos x}$.

$$f(x)=(\cos x)^{1/3}$$

$$f'(x)=\frac{1}{3}(\cos x)^{-2/3}\frac{d}{dx}(\cos x)$$

$$=\boxed{\frac{1}{3}(\cos x)^{-2/3}(-\sin x)}$$

Example 6: Find the derivative of $g(x)=\cos\sqrt[3]{x}$.

$$g(x)=\cos(x^{1/3})$$

$$g'(x)=-\sin(x^{\frac{1}{3}})\frac{d}{dx}(x^{\frac{1}{3}})$$

$$=-(\sin(x^{1/3}))(\tfrac{1}{3}x^{-2/3})$$

Example 7: Suppose that $y=\dfrac{1}{\cos x}$. Find $\dfrac{dy}{dx}$.

Note:

$$y=\frac{1}{\cos x}=\sec(x)$$

$$y=(\cos x)^{-1}$$

$$\frac{dy}{dx}=-1(\cos x)^{-2}\frac{d}{dx}(\cos x)$$

$$=-1(\cos x)^{-2}(-\sin x)=(\cos x)^{-2}(\sin x)$$

$$=\frac{\sin x}{(\cos x)^2}=\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}=\boxed{\sec x\tan x}$$

Ex $7\frac{1}{2}$: $f(x) = \tan(x^2+5)$

$$f'(x) = \sec^2(x^2+5)\frac{d}{dx}(x^2+5)$$
$$= \sec^2(x^2+5)(2x) = \boxed{2x\sec^2(x^2+5)}$$

Example 8: Suppose that $h(x) = (3x^2-4)^3(2x-9)^2$. Find $h'(x)$.

$$h(x) = \underbrace{(3x^2-4)^3}_{1st}\underbrace{(2x-9)^2}_{2nd}$$

$$h'(x) = (3x^2-4)^3\frac{d}{dx}(2x-9)^2 + (2x-9)^2\frac{d}{dx}(3x^2-4)^3$$
$$= (3x^2-4)^3\left[(2)(2x-9)\frac{d}{dx}(2x-9)\right] + (2x-9)^2\left[(3)(3x^2-4)^2\frac{d}{dx}(3x^2-4)\right]$$
$$= (3x^2-4)^3\left[2(2x-9)(2)\right] + (2x-9)^2\left[3(3x^2-4)^2(6x)\right]$$

~~**Example 9:** Suppose that $f(x) = \left(\frac{3x+1}{2x-1}\right)^5$. Find $f'(x)$.~~

$$= \underline{4(3x^2-4)^3(2x-9)} + \underline{18x(2x-9)^2(3x^2-4)^2}$$

(Factor out the GCF)

$$= 2(3x^2-4)^2(2x-9)\left[2(3x^2-4) + 9x(2x-9)\right]$$
$$= \underline{2(3x^2-4)^2(2x-9)}\underline{\left[6x^2-8+18x^2-81x\right]}$$
$$= \boxed{2(3x^2-4)^2(2x-9)(24x^2-81x-8)}$$

Example 10: Find the derivative of $y = \cos(\sin(\pi x^2))$.

$$\frac{dy}{dx} = -\sin(\sin(\pi x^2))\frac{d}{dx}(\sin(\pi x^2))$$
$$= -\sin(\sin(\pi x^2))\cos(\pi x^2)\frac{d}{dx}(\pi x^2)$$
$$= -\sin(\sin(\pi x^2))\cos(\pi x^2)(2\pi x)$$
$$= \boxed{-2\pi x\sin(\sin(\pi x^2))\cos(\pi x^2)}$$

Example 9: Suppose that $f(x)=\left(\frac{2x+1}{2x-1}\right)^5$. Find $f'(x)$.

$$f'(x)=5\left(\frac{2x+1}{2x-1}\right)^4\frac{d}{dx}\left(\frac{2x+1}{2x-1}\right)$$

$$=5\left(\frac{2x+1}{2x-1}\right)^4\left(\frac{(2x-1)(2)-(2x+1)(2)}{(2x-1)^2}\right)$$

$$=\frac{5(2x+1)^4}{(2x-1)^4}\left(\frac{\cancel{4x}-2-\cancel{4x}-2}{(2x-1)^2}\right)$$

$$=\frac{5(2x+1)^4}{(2x-1)^4}\left(\frac{-4}{(2x-1)^2}\right)=\boxed{\frac{-20(2x+1)^4}{(2x-1)^6}}$$

Another option: rewrite $f(x)=\left(\frac{2x+1}{2x-1}\right)^5$

as $f(x)=\frac{(2x+1)^5}{(2x-1)^5}$

Example 11: Find the derivative of $g(x) = \dfrac{\cos^2 x}{\sqrt{4x+1}}$.

See Archived notes, Spring 2015

for remaining examples :)

Example 12: Find the first and second derivatives of $f(x) = (x^2 + 4)^5$.

Example 13: Find the first and second derivatives of $f(x) = \cos\left(3x^2\right)$.

Example 14: Find the first and second derivatives of $y = \dfrac{2}{(3x+5)^2}$

Example 15: Find the equation of the tangent line to $y = \dfrac{x^2}{\sin \pi x + 1}$ at the point where $x = 1$.