2.5: Implicit Differentiation

Example 1: Given the equation $x^3 - 4y - 9x^2 = 5$, find $\frac{dy}{dx}$ by

a) Solving explicitly for y.
b) Implicit differentiation.

a)
$$x^3 - 4y - 9x^2 = 5$$
$$-4y = -x^3 + 9x^2 + 5$$
$$y = \frac{-x^3}{-4} + \frac{9x^2}{-4} + \frac{5}{-4}$$
$$y = \frac{x^3}{4} - \frac{9x^2}{4} - \frac{5}{4} = \frac{1}{4}x^3 - \frac{9}{4}x^2 - \frac{5}{4}$$
$$\frac{dy}{dx} = \frac{1}{4}(3x^2) - \frac{9}{4}(2x) - 0 = \boxed{\frac{3}{4}x^2 - \frac{9}{2}x}$$

b)
$$x^3 - 4y - 9x^2 = 5$$
$$\frac{d}{dx}(x^3 - 4y - 9x^2) = \frac{d}{dx}(5)$$
$$3x^2 - 4\frac{dy}{dx} - 18x = 0$$
Solve for $\frac{dy}{dx}$:
$$-4\frac{dy}{dx} = -3x^2 + 18x$$
$$\frac{dy}{dx} = \frac{-3x^2}{-4} + \frac{18x}{-4}$$
$$= \boxed{\frac{3}{4}x^2 - \frac{9}{2}x}$$

Example 2: Find $\frac{dy}{dx}$ for $xy = 4$.

$$\frac{d}{dx}(xy) = \frac{d}{dx}(4)$$

Product Rule:
$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = 0$$
$$x\frac{dy}{dx} + y(1) = 0$$
$$x\frac{dy}{dx} = -y$$
$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

This is the slope of the tangent line at the point (x, y)

Example 3: Find $\frac{dy}{dx}$ for the equation $x^3y - 2x^2y^3 + x^2 - 3 = 0$.

$$\frac{d}{dx}(x^3y - \underbrace{2x^2}_{1st}\underbrace{y^3}_{2nd} + x^2 - 3) = \frac{d}{dx}(0)$$
$$x^3\frac{d}{dx}(y) + y\frac{d}{dx}(x^3) - \left[2x^2\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(2x^2)\right] + 2x - 0 = 0$$
$$x^3\frac{dy}{dx} + y(3x^2) - \left[2x^2 \cdot 3y^2\frac{dy}{dx} + y^3(4x)\right] + 2x = 0$$
$$x^3\frac{dy}{dx} + 3x^2y - 6x^2y^2\frac{dy}{dx} - 4xy^3 + 2x = 0$$
$$x^3\frac{dy}{dx} - 6x^2y^2\frac{dy}{dx} = -3x^2y + 4xy^3 - 2x$$
$$\frac{dy}{dx}(x^3 - 6x^2y^2) = -3x^2y + 4xy^3 - 2x \Longrightarrow \boxed{\frac{dy}{dx} = \frac{-3x^2y + 4xy^3 - 2x}{x^3 - 6x^2y^2}}$$

Note: this is equivalent to:
$$\frac{dy}{dx} = \frac{3x^2y - 4xy^3 + 2x}{6x^2y^2 - x^3}$$
(Multiply by $\frac{-1}{-1}$)

Ex $1\frac{1}{4}$: $x^3 + y^3 = 7$. Find $\frac{dy}{dx}$

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(7)$$

$$3x^2 + 3y^2 \frac{d}{dx}(y) = 0$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

Note:

$$\frac{d}{dx}(\sin x)^3$$

$$= 3(\sin x)^2 \frac{d}{dx}(\sin x)$$

$$= 3(\sin x)^2(\cos x)$$

we treat the y like the $\sin x$ above

solve for $\frac{dy}{dx}$:

$$3y^2 \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = -\frac{x^2}{y^2}$$

$$\boxed{\frac{dy}{dx} = -\frac{x^2}{y^2}}$$

Ex $1\frac{1}{2}$: $3x^4 + 2y^4 - 5x^2 + 7y = y^2 + y^5 - x^3$

$$\frac{d}{dx}(3x^4 + 2y^4 - 5x^2 + 7y) = \frac{d}{dx}(y^2 + y^5 - x^3)$$

$$12x^3 + 8y^3\frac{dy}{dx} - 10x + 7\frac{dy}{dx} = 2y\frac{dy}{dx} + 5y^4\frac{dy}{dx} - 3x^2$$

Get all terms with $\frac{dy}{dx}$ on one side;
all terms without $\frac{dy}{dx}$ on other side:

$$8y^3\frac{dy}{dx} + \frac{7dy}{dx} - 2y\frac{dy}{dx} - 5y^4\frac{dy}{dx} = -3x^2 - 12x^3 + 10x$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}(8y^3 + 7 - 2y - 5y^4) = -3x^2 - 12x^3 + 10x$$

$$\boxed{\frac{dy}{dx} = \frac{-3x^2 - 12x^3 + 10x}{8y^3 + 7 - 2y - 5y^4}}$$

Example 4: Find $\frac{dy}{dx}$ for the equation $\frac{1}{y^4} - \frac{5}{x^4} = 1$.

Rewrite: $y^{-4} - 5x^{-4} = 1$

$$\frac{d}{dx}\left(y^{-4} - 5x^{-4}\right) = \frac{d}{dx}(1)$$

$$-4y^{-5}\frac{d}{dx}(y) + 20x^{-5} = 0$$

Solve for $\frac{dy}{dx}$: $\quad -\frac{4}{y^5}\cdot\frac{dy}{dx} = -\frac{20}{x^5}$

$$-\frac{y^5}{4}\left(-\frac{4}{y^5}\cdot\frac{dy}{dx}\right) = \left(-\frac{20}{x^5}\right)\left(-\frac{y^5}{4}\right)$$

$$\frac{dy}{dx} = \frac{20y^5}{4x^5} = \boxed{\frac{5y^5}{x^5}}$$

Example 5: Find $\frac{dy}{dx}$ for the equation $(x-y)^4 = y^2$.

$$\frac{d}{dx}(x-y)^4 = \frac{d}{dx}(y^2)$$

$$4(x-y)^3\frac{d}{dx}(x-y) = 2y\frac{dy}{dx}$$

$$4(x-y)^3\left(1 - \frac{dy}{dx}\right) = 2y\frac{dy}{dx}$$

$$4(x-y)^3 - 4(x-y)^3\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Example 6: Find $\frac{dy}{dx}$ for the equation $x + \cos($[redacted]$y) = y$

$$x + \cos(y) = y$$

$$\frac{d}{dx}(x + \cos(y)) = \frac{d}{dx}(y)$$

$$1 - \sin(y)\frac{dy}{dx} = \frac{dy}{dx}$$

$$1 = \frac{dy}{dx} + \sin(y)\frac{dy}{dx}$$

$$1 = \frac{dy}{dx}(1 + \sin y)$$

$$\boxed{\frac{1}{1+\sin y} = \frac{dy}{dx}}$$

Finding higher-order derivatives using implicit differentiation:

To find the second derivative, denoted $\frac{d^2y}{dx^2}$, differentiate the first derivative $\frac{dy}{dx}$ with respect to *x*.

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example 7: Find $\frac{d^2y}{dx^2}$ for the equation $x^3 - 2x^2 = y$. (find 2nd derivative)

Find 1st derivative: $\frac{d}{dx}(x^3 - 2x^2) = \frac{d}{dx}(y)$

$3x^2 - 4x = \frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 4x) = \boxed{6x - 4}$$

Example 8: Find $\frac{d^2y}{dx^2}$ for the equation $xy^2 - y = 3$.

Ex $7\frac{1}{2}$: $\cos(x^2+5y) = \tan(6x)$ Find $\frac{d^2y}{dx^2}$.

$$\frac{d}{dx}\left(\cos(x^2+5y)\right) = \frac{d}{dx}\left(\tan(6x)\right)$$

$$-\sin(x^2+5y)\frac{d}{dx}(x^2+5y) = \sec^2(6x)\underbrace{\frac{d}{dx}(6x)}_{6}$$

$$-\sin(x^2+5y)\left(2x+5\frac{dy}{dx}\right) = 6\sec^2(6x)$$

$$-2x\sin(x^2+5y) - 5\frac{dy}{dx}\sin(x^2+5y) = 6\sec^2(6x)$$

$$-5\frac{dy}{dx}\sin(x^2+5y) = 6\sec^2(6x) + 2x\sin(x^2+5y)$$

$$\frac{dy}{dx} = \frac{6\sec^2(6x)}{-5\sin(x^2+5y)} - \frac{2x}{5}$$

$$\frac{d^2y}{dx^2} = \frac{-5\sin(x^2+5y)\frac{d}{dx}\left(6\sec^2 6x\right) - 6\sec^2(6x)\frac{d}{dx}\left(-5\sin(x^2+5y)\right)}{\left(-5\sin(x^2+5y)\right)^2} - \frac{2}{5}$$

$$= \frac{-5\sin(x^2+5y)\,12\sec(6x)\sec(6x)\tan(6x)(6) - 6\sec^2(6x)\left(-5\cos(x^2+5y)\right)\left(2x+5\frac{dy}{dx}\right)}{25\sin^2(x^2+5y)}$$

$$-\frac{2}{5}$$

Ex: $\cos(x) + \sin(y) = x^2$. Find $\frac{d^2y}{dx^2}$

$$\frac{d}{dx}\left(\cos(x) + \sin(y)\right) = \frac{d}{dx}(x^2)$$

$$-\sin(x) + \cos(y)\frac{dy}{dx} = 2x$$

$$\cos(y)\frac{dy}{dx} = 2x + \sin x$$

$$\frac{dy}{dx} = \frac{2x+\sin x}{\cos y}$$

$$\frac{d^2y}{dx^2} = \frac{\cos y\frac{d}{dx}(2x+\sin x) - (2x+\sin x)\frac{d}{dx}(\cos y)}{(\cos y)^2}$$

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$$\frac{d^2y}{dx^2} = \frac{\cos y\,(2+\cos x) - (2x+\sin x)\left(-\sin y\frac{dy}{dx}\right)}{\cos^2 y}$$

$$= \frac{2\cos y + \cos y\cos x + \sin y\frac{dy}{dx}(2x+\sin x)}{\cos^2 y}$$

substitute

$$\frac{dy}{dx} = \frac{2x+\sin x}{\cos y} \Rightarrow \frac{2\cos y + \cos y\cos x + \sin y\left(\frac{2x+\sin x}{\cos y}\right)(2x+\sin x)}{\cos^2 y}$$

$$= \frac{2\cos y + \cos y\cos x + \tan y\,(2x+\sin x)^2}{\cos^2 y}$$

Example 9: Find the equation of the tangent line to the ellipse $4x^2+16y^2=64$ at the point $(2,\sqrt{3})$.

$$\frac{d}{dx}\left(4x^2+16y^2\right)=\frac{d}{dx}(64)$$

$$8x+32y\frac{dy}{dx}=0$$

$$32y\frac{dy}{dx}=-8x$$

$$\frac{dy}{dx}=-\frac{8x}{32y}=-\frac{x}{4y}$$

$$m=\left.\frac{dy}{dx}\right|_{x=2,\,y=\sqrt{3}}=\left.-\frac{x}{4y}\right|_{x=2,\,y=\sqrt{3}}=-\frac{2}{4\sqrt{3}}=-\frac{1}{2\sqrt{3}}=-\frac{\sqrt{3}}{6}$$

$$y-y_1=m(x-x_1)$$

$$y-\sqrt{3}=-\frac{\sqrt{3}}{6}(x-2)$$

$$y=-\frac{\sqrt{3}}{6}x+\frac{2\sqrt{3}}{6}+\sqrt{3}\left(\frac{6}{6}\right)$$

$$y=-\frac{\sqrt{3}}{6}x+\frac{8\sqrt{3}}{6}$$

$$\boxed{y=-\frac{\sqrt{3}}{6}x+\frac{4\sqrt{3}}{6}}$$

Definition: Two curves are said to be *orthogonal* if at each point of intersection, their tangent lines are perpendicular.

Example 10: Show that the hyperbola $x^2-y^2=5$ and the ellipse $4x^2+9y^2=72$ are orthogonal.