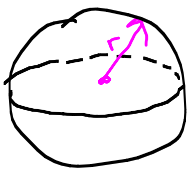
2.6: Related rates

General idea for solving rate problems:

1. Draw a sketch if applicable. The only dimensions you put on your sketch should be those that do not change.
2. Write down, in calculus notation, the rates you know and want.
3. Write an equation relating the quantities that are changing.
4. Differentiate it implicitly, with respect to time.
5. Substitute known quantities.
6. Solve for the required rate.

Example 1: The radius of a sphere is increasing at the rate of 2 inches per minute. Find the rate of change in volume when the radius is 6 inches.



Know: $\frac{dr}{dt} = +2 \text{ in/min}$

Want: $\frac{dV}{dt}$ (V = volume)

When: $r = 6$ in

write an equation that relates r to V:

$V = \frac{4}{3}\pi r^3$

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$

$\frac{dV}{dt} = \frac{4}{3}\pi \frac{d}{dt}(r^3)$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

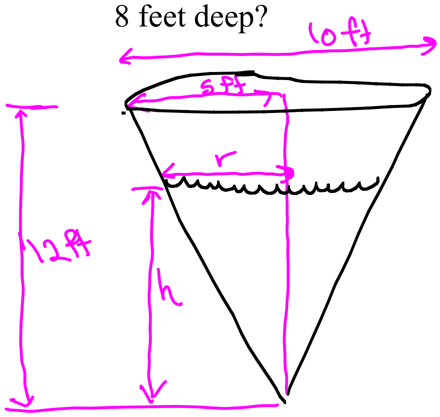
Substitute $r = 6\text{in}$, $\frac{dr}{dt} = \frac{2\text{in}}{\text{min}}$:

$\frac{dV}{dt} = 4\pi (6\text{in})^2 \left(\frac{2\text{in}}{\text{min}}\right)$

$= \boxed{288\pi \text{ in}^3/\text{min}}$

$\approx 904.78 \text{ in}^3/\text{min}$

Example 2: A conical tank (with vertex down) has a diameter of 10 feet at the top and is 12 feet deep. If water is draining out at 10 cubic feet per minute, what is the rate of change in depth when the water is 8 feet deep?



Know: $\frac{dV}{dt} = -\frac{10 \text{ ft}^3}{\text{min}}$ (V = volume of water)

Want: $\frac{dh}{dt}$

When: $h = 8$ ft

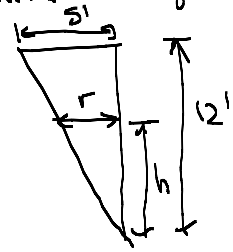
Recall: Volume of cone is

$V = \frac{1}{3}\pi r^2 h$

we want V and h only ... need to get rid of r.

Need a relationship between r and h.

Similar triangles:



$\frac{r}{h} = \frac{5}{12}$

$r = \frac{5}{12}h \Leftarrow$ or $\frac{r}{5} = \frac{h}{12}$

see next page

Ex 2 cont'd:

$$V = \frac{1}{3}\pi r^2 h$$

Substitute $r = \frac{5}{12}h$

$$V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{25}{144}h^2\right)h$$

$$V = \frac{25\pi}{432}h^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{25\pi}{432}h^3\right)$$

$$\frac{dV}{dt} = \frac{25\pi}{432} \cdot 3h^2 \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{75\pi h^2}{432}\frac{dh}{dt}$$

$h = 8\text{ ft}$, $\frac{dV}{dt} = -10\text{ ft}^3/\text{min}$

$$\Rightarrow -\frac{10\text{ ft}^3}{\text{min}} = \frac{75\pi}{432}(8\text{ft})^2\frac{dh}{dt}$$

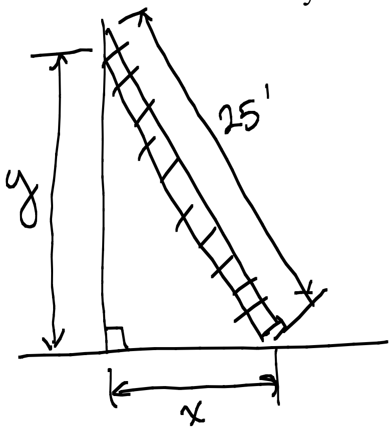
$$-\frac{10\text{ ft}^3}{\text{min}} = \frac{4800\pi\text{ ft}^2}{432} \cdot \frac{dh}{dt}$$

$$-\frac{10\text{ ft}^3}{\text{min}} \cdot \frac{432}{4800\pi\text{ ft}^2} = \frac{dh}{dt}$$

$$-\frac{0.9}{\pi}\text{ ft/min} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = \boxed{-\frac{9}{10\pi}\text{ ft/min} \approx -0.286\text{ ft/min}}$$

Example 3: A ladder 25 feet long is leaning against a wall. The base of the ladder is pulled away at 2 feet per second. How fast is the top of the ladder moving when the base is 9 feet away? What about when it is 24 feet away?



Know: $\frac{dx}{dt} = +2$ ft/sec

Want: $\frac{dy}{dt}$

When: $x = 24'$, $x = 9'$

Need eqn relating x (from the rate I know) to y (from the rate I want)

$$x^2 + y^2 = (25\text{ ft})^2$$

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(625\text{ ft}^2)$$

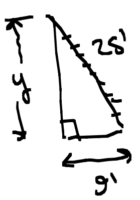
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

See next page

$$2y\frac{dy}{dt} = -2x\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{-2x}{2y}\cdot\frac{dx}{dt} = -\frac{x}{y}\cdot\frac{dx}{dt}$$

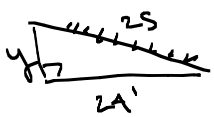
Need to find y for $x = 9'$ and $x = 24'$:



$y^2 + 9^2 = 25^2$

$y^2 = 625 - 81 = 544$

$y = \sqrt{544}$ ft



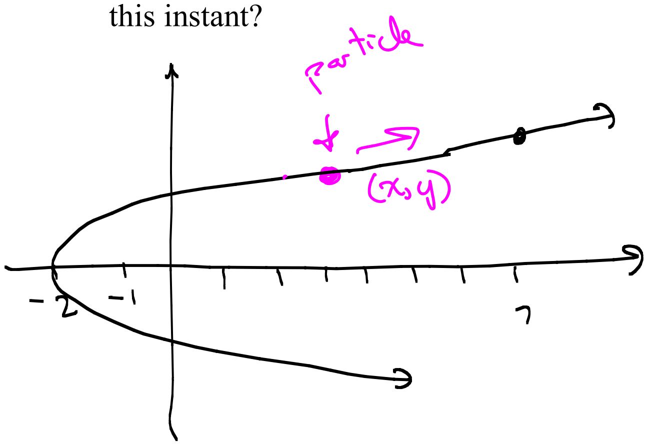
$y^2 + (24\text{ ft})^2 = (25\text{ ft})^2$

$y^2 + 576\text{ ft}^2 = 625\text{ ft}^2$

$y^2 = 49\text{ ft}^2 \Rightarrow y = \sqrt{49}\text{ ft} = 7\text{ ft}$

$$\left.\frac{dy}{dt}\right|_{x=9',\ y=\sqrt{544}\text{ ft},\ dx/dt = 2\text{ ft/sec}} = -\frac{9\text{ ft}}{\sqrt{544}\text{ ft}}\cdot 2\text{ ft/sec} = \boxed{-\frac{18}{\sqrt{544}}\text{ ft/sec when } x = 9\text{ ft}}$$

Example 4: A particle is moving along the parabola $y^2 = 4x + 8$. As it passes through the point (7, 6) its y-coordinate is increasing at the rate of 3 units per second. How fast is the x-coordinate changing at this instant?



$y^2 = 4x + 8$

$y^2 = 4(x+2)$

shift it left 2

Know: $\frac{dy}{dt} = +3$ units/sec

Want: $\frac{dx}{dt}$

When: $x = 7$, $y = 6$

Relationship between y and x: $y^2 = 4x + 8$

$$2y\frac{dy}{dt} = 4\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{2y\frac{dy}{dt}}{4} = \frac{2(6)(3\text{ units/sec})}{4} = \boxed{9\text{ units/sec}}$$

Ex 3 cont'd

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$\left.\frac{dy}{dt}\right|_{\substack{x = 9\text{ ft} \\ y = \sqrt{544}\text{ ft} \\ \frac{dx}{dt} = 2\text{ ft/sec}}} = -\frac{9\text{ ft}}{\sqrt{544}\text{ ft}} \cdot 2\text{ ft/sec} = -\frac{18}{\sqrt{544}}\text{ ft/sec}$$

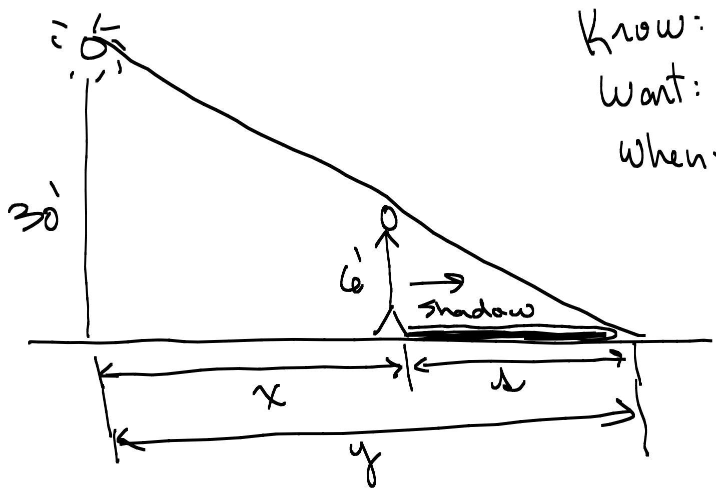
$$\approx \boxed{-0.7717\text{ ft/sec}}$$

when $x = 9$ ft

$$\left.\frac{dy}{dt}\right|_{\substack{x = 24\text{ ft} \\ y = 7\text{ ft} \\ \frac{dx}{dt} = 2\text{ ft/sec}}} = -\frac{24\cancel{\text{ft}}}{7\cancel{\text{ft}}} \cdot 2\text{ ft/sec}$$

$$= -\frac{48}{7}\text{ ft/sec} = \boxed{-6\tfrac{6}{7}\text{ ft/sec}}$$

Example 5: A 6-foot tall man walks away from a 30-foot tall lamppost at a speed of 400 feet per minute. When he is 50 feet away from the lamppost, at what rate is his shadow lengthening? How fast is the tip of his shadow moving?



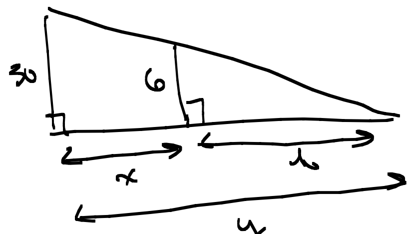
Know: $\frac{dx}{dt} = +400$ ft/min

Want: $\frac{ds}{dt}$, $\frac{dy}{dt}$

When: $x = 50$ ft

Need a relationship between x (from rate we know) and either s or y (from rates we want)

Similar Triangles:



$\frac{s}{6} = \frac{y}{30}$ or $\frac{6}{30} = \frac{s}{y}$

substitute $y = x + s$ or $s = y - x$

$y = x + s$ substitute

$\frac{s}{6} = \frac{x+s}{30}$

$\frac{30s}{6} = x + s$

$5s = x + s$

$4s = x$

$\frac{d}{dt}(4s) = \frac{d}{dt}(x)$

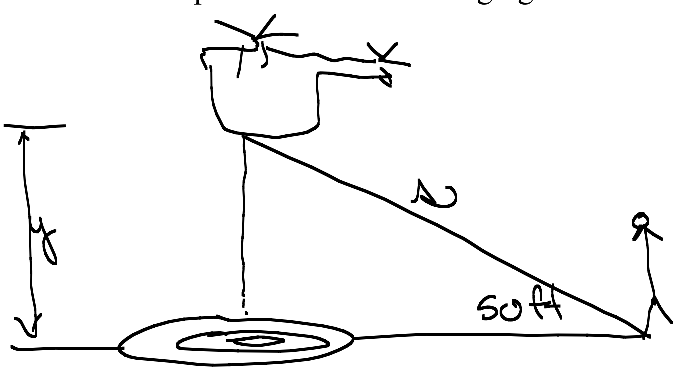
$4\frac{ds}{dt} = \frac{dx}{dt}$

$\frac{ds}{dt} = \frac{1}{4}\frac{dx}{dt} = \frac{1}{4}\left(400 \frac{ft}{min}\right)$

$= 100$ ft/min

Shadow is lengthening at 100 ft/min

Example 6: At a distance of 50 ft from the pad, a man observes a helicopter taking off from a heliport. The helicopter is rising vertically at a speed of 44 ft/second. How fast is the distance between the helicopter and the man changing when it is at an altitude of 120 ft?



Know: $\frac{dy}{dt} = +\frac{44 ft}{sec}$

Want: $\frac{ds}{dt}$

When: $y = 120'$

$y^2 + (50\text{ ft})^2 = s^2$

$2y\frac{dy}{dt} + 0 = 2s\frac{ds}{dt}$

$2(120\text{ ft})\frac{44\text{ ft}}{sec} = 2(130\text{ ft})\frac{ds}{dt}$

At moment when $y = 120'$

$50^2 + 120^2 = s^2$

$s^2 = 16900$

$s = 130$

$\frac{\cancel{2}(120\,\cancel{ft})(44\text{ ft/sec})}{\cancel{2}(130\,\cancel{ft})} = \frac{ds}{dt}$

$\frac{ds}{dt} = \frac{120}{130}(44)$ ft/sec $\approx \boxed{40.615 \text{ ft/sec}}$

Ex 5 cont'd

Now find $\frac{dy}{dt}$:

use $y = x + s$

$\frac{d}{dt}(y) = \frac{d}{dt}(x+s)$

$\frac{dy}{dt} = \frac{dx}{dt} + \frac{ds}{dt}$

$= \frac{400 ft}{min} + \frac{100 ft}{min}$

$= \boxed{500 \text{ ft/min}}$