3.5: Limits at Infinity

There are two types of limits involving infinity.

$\lim_{x \to \infty} f(x)$ $\lim_{x \to -\infty} f(x)$

Limits at infinity, written in the form $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$, are related to horizontal asymptotes.

Infinite limits (covered in Section 1.5) take the form of statements like $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$. Infinite limits generally result in vertical asymptotes.

When combined, these two types of limits involving infinity result in statements such as $\lim_{x \to \infty} f(x) = \infty$ or $\lim_{x \to \infty} f(x) = -\infty$, which describe the end behavior of graphs.

Limits at infinity:

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

More precisely, $\lim_{x \to \infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number $M > 0$ such that for all x, $|f(x) - L| < \varepsilon$ whenever $x > M$.

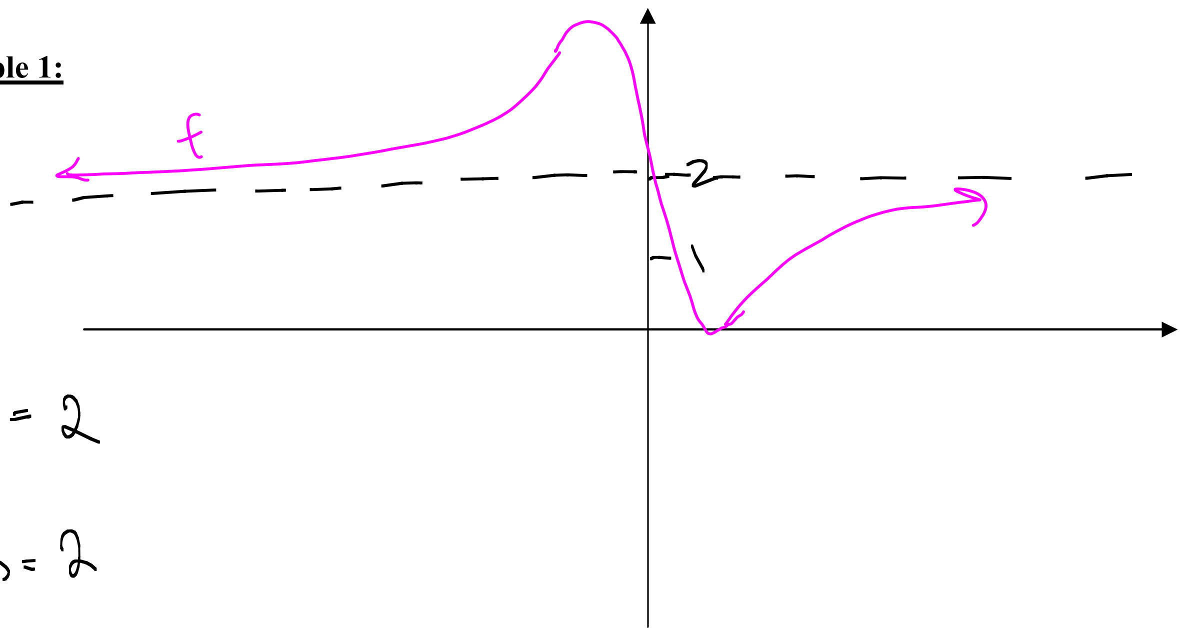
Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by making x a sufficiently large negative number.

More precisely, $\lim_{x \to \infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number $N < 0$ such that for all x, $|f(x) - L| < \varepsilon$ whenever $x < N$.

Example 1:



$\lim_{x \to \infty} f(x) = 2$

$\lim_{x \to -\infty} f(x) = 2$

Horizontal asymptotes:

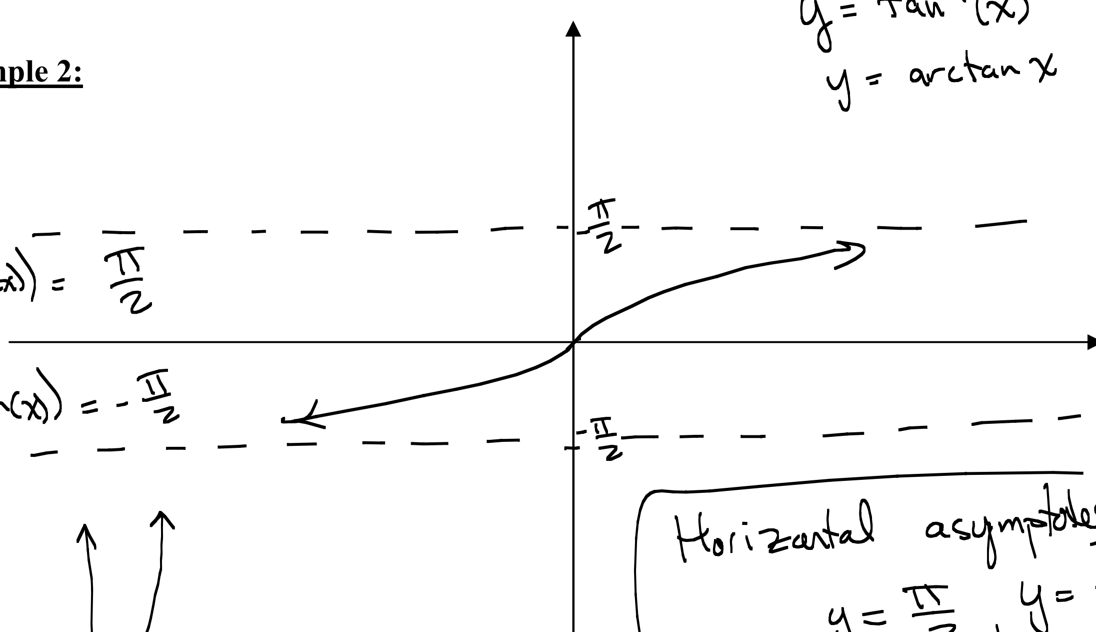
The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L .$$

Example 2:

$y = \tan^{-1}(x)$

$y = \arctan x$



$\lim_{x \to \infty} (\arctan(x)) = \frac{\pi}{2}$

$\lim_{x \to -\infty} (\arctan(x)) = -\frac{\pi}{2}$

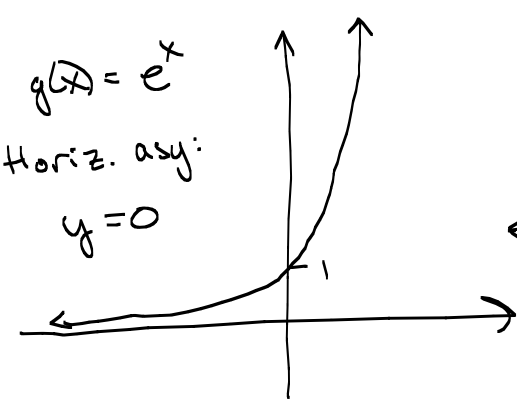
Horizontal asymptotes

$y = \frac{\pi}{2}, \ y = -\frac{\pi}{2}$

$g(x) = e^x$

Horiz. asy:

$y = 0$



Here, $\lim_{x \to \infty} e^x = +\infty$, $\lim_{x \to -\infty} e^x = 0$

Example 3: Determine $\lim_{x\to\infty}\frac{1}{x}$ and $\lim_{x\to-\infty}\frac{1}{x}$.

As $x \to +\infty$, $\frac{1}{x} \to \frac{1}{+\text{huge}} \to +\text{tiny} \to 0$

so $\boxed{\lim_{x\to\infty}\frac{1}{x} = 0}$

As $x \to -\infty$, $\frac{1}{x} \to \frac{1}{-\text{huge}} \to -\text{tiny} \to 0$

so $\boxed{\lim_{x\to-\infty}\frac{1}{x} = 0}$

Theorem: If $r > 0$ is a rational number, then $\lim_{x\to\infty}\frac{1}{x^r} = 0$. Also $\lim_{x\to-\infty}\frac{1}{x^r} = 0$ if x^r is defined for all x.

$\lim_{x\to\infty}\frac{1}{x^r} = 0$ also $\lim_{x\to-\infty}\frac{1}{x^r} = 0$ if x^r is defined for all x

Example 4: Determine $\lim_{x\to\infty}\left(9+\frac{3}{x^4}\right)$

$$\lim_{x\to\infty}\left(9+\frac{3}{x^4}\right) = \lim_{x\to\infty} 9 + 3\lim_{x\to\infty}\frac{1}{x^4} = 9 + 3(0) = \boxed{9}$$

To take advantage of this theorem when determining the limit at infinity of a quotient, divide numerator and denominator by the largest power of the variable in the denominator.

Example 5: Find $\lim_{x\to\infty}\frac{3-2x}{4x+6}$.

$$\lim_{x\to\infty}\frac{3-2x}{4x+6} = \lim_{x\to\infty}\frac{3-2x}{4x+6}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) = \lim_{x\to\infty}\left(\frac{\frac{3}{x}-\frac{2x}{x}}{\frac{4x}{x}+\frac{6}{x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\frac{3}{x}-2}{4+\frac{6}{x}}\right) = \frac{0-2}{4+0} = \frac{-2}{4} = \boxed{-\frac{1}{2}}$$

Example 6: Find $\lim_{x\to\infty}\frac{2x^2+15x+9}{5x^3-14}$ and $\lim_{x\to-\infty}\frac{2x^2+15x+9}{5x^3-14}$.

$$\lim_{x\to\pm\infty}\frac{2x^2+15x+9}{5x^3-14}=\lim_{x\to\pm\infty}\frac{\frac{2x^2}{x^3}+\frac{15x}{x^3}+\frac{9}{x^3}}{\frac{5x^3}{x^3}-\frac{14}{x^3}}=\lim_{x\to\pm\infty}\frac{\frac{2}{x}+\frac{15}{x^2}+\frac{9}{x^3}}{5-\frac{14}{x^3}}$$

$$=\frac{0+0+0}{5-0}=\frac{0}{5}=\boxed{0}$$

Horizontal asymptote for $f(x)=\frac{2x^2+15x+9}{5x^3-14}$ is $\boxed{y=0}$

Example 7: Find the horizontal asymptote (if any) of $h(x)=\frac{8x^2-6x+1}{3x^2+4x-5}$.

$$\lim_{x\to\pm\infty}\frac{8x^2-6x+1}{3x^2+4x-5}=\lim_{x\to\pm\infty}\frac{\frac{8x^2}{x^2}-\frac{6x}{x^2}+\frac{1}{x^2}}{\frac{3x^2}{x^2}+\frac{4x}{x^2}-\frac{5}{x^2}}=\lim_{x\to\pm\infty}\frac{8-\frac{6}{x}+\frac{1}{x^2}}{3+\frac{4}{x}-\frac{5}{x^2}}$$

$$=\frac{8-0+0}{3+0-0}=\boxed{\frac{8}{3}}$$

h has horizontal asymptote $\boxed{y=\frac{8}{3}}$

$\boxed{f(x)=\frac{7x^5-5x+1}{8-x^2} \text{ does not have a horizontal asymptote.}}$

Example 8: Determine $\lim_{x\to\infty}\frac{7x^5-5x+1}{8-x^2}$ and $\lim_{x\to-\infty}\frac{7x^5-5x+1}{8-x^2}$.

$$\lim_{x\to\infty}\frac{7x^5-5x+1}{8-x^2}=\lim_{x\to\infty}\frac{\frac{7x^5}{x^2}-\frac{5x}{x^2}+\frac{1}{x^2}}{\frac{8}{x^2}-\frac{x^2}{x^2}}=\lim_{x\to\infty}\frac{7x^3-\frac{5}{x}+\frac{1}{x^2}}{\frac{8}{x^2}-1}$$

$$=\frac{\lim_{x\to\infty}(7x^3)-0+0}{0-1}=\frac{7\lim_{x\to\infty}(x^3)}{-1}=-7\lim_{x\to\infty}(x^3)=\boxed{-\infty}$$

as $x\to\infty$, $y\to-7(+\text{huge})^3\to-\text{huge}$

$$\lim_{x\to-\infty}\frac{7x^5-5x+1}{8-x^2}=\lim_{x\to-\infty}(-7x^3)=\boxed{\infty}$$

(these limits don't exist)

as $x\to-\infty$, $y\to-7(-\text{huge})^3\to-7(-\text{huge})\to+\text{huge}$

Ex.: Find horizontal asymptotes (if any) of

$$y = \frac{\cos x}{x}.$$

Horizontal asymptote: $y = 0$

$$\lim_{x \to \pm\infty} \left(\frac{\cos(x)}{x} \right) = \lim_{x \to \pm\infty} \left(\frac{1}{x} \right) = 0$$

(because $-1 \leq \cos x \leq 1$)

$$= \lim_{x\to\infty} \frac{x^{\frac{1}{3}}\left(1-x^{\frac{1}{6}}\right)}{8} = \boxed{-\infty}$$

Note:
$\infty - \infty$ is an indeterminate forms

Also $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are indeterminate forms

Example 9: Determine $\lim\limits_{x\to\infty} \dfrac{\sqrt[3]{x}-\sqrt{x}}{8-x^{-2}}$

$$\lim_{x\to\infty} \frac{\sqrt[3]{x}-\sqrt{x}}{8-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^{\frac{1}{3}}-x^{\frac{1}{2}}}{8-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^{\frac{1}{3}}\left(1-x^{\frac{1}{2}-\frac{1}{3}}\right)}{8-0}$$

as $x\to\infty$, $y \longrightarrow \dfrac{\sqrt[3]{+\text{huge}}-\sqrt{+\text{huge}}}{8-\frac{1}{(+\text{huge})^2}} \to \dfrac{+\text{huge}-+\text{huge}}{8}$

$$y \longrightarrow \frac{(+\text{huge})^{1/3}\left(1-(+\text{huge})^{1/6}\right)}{8} \longrightarrow \frac{\sqrt[3]{\text{huge}}\left(1-\sqrt[6]{\text{huge}}\right)}{8} \to +\text{huge}(1-\text{huge}) \to +\text{huge}(-\text{huge}) \to -\text{huge}$$

Example 10: Evaluate the limit of $f(x) = \dfrac{x^3+4x^2-11x-7}{x^2+6x+1}$ as x approaches $\pm\infty$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+4-\frac{11}{x}-\frac{7}{x^2}}{1+\frac{6}{x}+\frac{1}{x^2}} = \frac{\left(\lim\limits_{x\to\infty} x\right)+4-0-0}{1+0+0}$$

$$= \lim_{x\to\infty} (x)+4$$

$$= \lim_{x\to\infty} x = \boxed{\infty}$$

similarly,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x = \boxed{-\infty}$$

because the degree of the numerator is 1 more than the degree of the denominator, f has a slant (oblique) asymptote.

See next page

Example 11: Evaluate the limit of $f(x) = \dfrac{8x^2-7x+1}{2x-3}$ as x approaches $\pm\infty$.

Find the slant asymptote:

$\frac{8x^2}{2x} \to 4x$, $\frac{5x}{2x} \to \frac{5}{2}$

$$\begin{array}{r} 4x + \frac{5}{2} \\ 2x-3 \,\big)\, 8x^2-7x+1 \\ -(8x^2-12x) \\ \hline 5x+1 \\ -\left(5x-\frac{15}{2}\right) \\ \hline 1+\frac{15}{2}=\frac{17}{2} \end{array}$$

$$\frac{5}{2}(2x-3) = 5x - \frac{15}{2}$$

$$f(x) = \frac{8x^2-7x+1}{2x-3}$$

$$= 4x+\frac{5}{2} + \frac{17/2}{2x-3}$$

As $x\to\pm\infty$,
$f(x)$ approaches $y = 4x+\frac{5}{2}$

Slant asymptote: $\boxed{y = 4x+\frac{5}{2}}$

Ex 10: Find the slant (oblique) asymptote:

$$f(x) = \frac{x^3+4x^2-11x-7}{x^2+6x+1}$$

Long division:

$$\begin{array}{r} x-2 \\ x^2+6x+1 \overline{\big)\, x^3+4x^2-11x-7} \\ -(x^3+6x^2+x) \\ \hline -2x^2-12x-7 \\ -(-2x^2-12x-2) \\ \hline -5 \end{array}$$

Check: $(x-2)(x^2+6x+1)-5$

$= x^3+6x^2+x$
$\quad -2x^2-12x-2-5$

$= x^3+4x^2-11x-7$ ✓

So, $f(x) = \dfrac{x^3+4x^2-11x-7}{x^2+6x+1} = x-2+\dfrac{-5}{x^2+6x+1}$

Note: $\displaystyle\lim_{x\to\pm\infty} \frac{-5}{x^2+6x+1} = \lim_{x\to\pm\infty} \frac{-\frac{5}{x^2}}{1+\frac{6}{x}+\frac{1}{x^2}} = \frac{-0}{1+0+0}$
(divide by x^2) $\quad = 0$

As $x \to \pm\infty$, $f(x)$ looks more and more like $g(x) = x-2$

Slant asymptote: $\boxed{y = x-2}$

Ex: $f(x) = 8x$

Slant asymptotes:

> Note:
> The graphs of the functions in the previous two examples have *oblique (slant) asymptotes*. This is because the function values (y-values) approached those of a linear function $y = mx + b$ as x approached $\pm\infty$.

Example 12: Evaluate the limit of $f(x) = \dfrac{2x^3 - x^2 + x}{x-3}$ as x approaches $\pm\infty$. Does the graph of this function have a slant asymptote?

$$
\begin{array}{r|l}
 & 2x^2 + 5x + 16 \\
x-3 & 2x^3 - x^2 + x + 0 \\
 & -(2x^3 - 6x^2) \\
 & 5x^2 + x \\
 & -(5x^2 - 15x) \\
 & 16x + 0 \\
 & -(16x - 48) \\
 & 48
\end{array}
$$

$f(x) = 2x^2 + 5x + 16 + \dfrac{48}{x-3}$

As $x \to \pm\infty$, the graph of x approach the parabola $y = 2x^2 + 5x + 6$