4.1: Antiderivatives and Indefinite Integration

Definition: An *antiderivative* of *f* is a function whose derivative is *f*.

i.e. A function *F* is an antiderivative of *f* if $F'(x) = f(x)$.

Example 1: $F(x) = x^3 + 5x$ is an antiderivative of $f(x) = 3x^2 + 5$.

What are some more antiderivatives of $f(x) = 3x^2 + 5$?

$G(x) = x^3 + 5x + 7$

$H(x) = x^3 + 5x - \frac{13\pi}{12}$

If C is any constant, then $F(x) = x^3 + 5x + C$ is an antiderivative of $f(x) = 3x^2 + 5$

So we have a whole "family" of antiderivatives of *f*.

Definition: A function *F* is called an antiderivative of *f* on an interval *I* if $F'(x) = f(x)$ for all *x* in *I*.

Theorem: If *F* is an antiderivative of *f* on an interval *I*, then all antiderivatives of *f* on *I* will be of the form $F(x) + C$ where *C* is an arbitrary constant.

Example 2: Find the general form of the antiderivatives of $f(x) = 3x^2 + 5$.

$F(x) = x^3 + 5x + C$

Example 3: Find the general form of the antiderivatives of $f(x) = 6x^5 + \cos x$.

$F(x) = x^6 + \sin x + C$

Check: $F'(x) = 6x^5 + \cos x + 0 = f(x)$

Integration:

Integration is the process of finding antiderivatives.

$\int f(x)\,dx$

> $\int f(x)dx$ is called the *indefinite integral* of *f*.
>
> $\int f(x)dx$ is the family of antiderivatives, or the most general antiderivative of *f*.
>
> This means: $\int f(x)dx = F(x) + c$, where $F'(x) = f(x)$.
>
> The *c* is called the *constant of integration*.

Integral sign

"with respect to *x*"

$$\int f(x)dx$$

Integrand

From 3.9

If $y = f(x)$ is differentiable, then dx represents an amount of change in x

$dx \approx \Delta x$

The differential dy is defined to be $dy = f'(x)\,dx$

Note: this gives you $\frac{dy}{dx} = f'(x)$

Example 4: Find $\int 3x^2 + 5\, dx$.

$\int (3x^2+5)dx = \boxed{x^3 + 5x + c}$

Example 5: Find $\int 6x^5 + \cos x\, dx$.

$\int (6x^5 + \cos x)\, dx = \boxed{x^6 + \sin x + C}$

Example 6: Find $\int \sec^2 x\, dx$.

$\int \sec^2 x\, dx = \boxed{\tan x + c}$

because $\frac{d}{dx}(\tan x + c) = \sec^2 x$

Rules for Finding Antiderivatives:

Notation in this table: F is an antiderivative of f, G is an antiderivative of g,

Function	Antiderivative
k	$kx+c$
$kf(x)$	$kF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n for $n \neq -1$	$\dfrac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

(k is a constant)

$\frac{d}{dx}(-\cos x) = -\frac{d}{dx}(\cos x) = -(-\sin x) = \sin x$

1. $\int k\ dx = kx + c$ (k a constant)

2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$ $(n \neq -1)$ (Power rule)

3. $\int k\ f(x)\ dx = k\int f(x)\ dx$ (k a constant)

4. $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

5. $\int \cos x\, dx = \sin x + c$

6. $\int \sin x\, dx = -\cos x + c$

7. $\int \sec^2 x\, dx = \tan x + c$ because $\frac{d}{dx}(\tan x) = \sec^2 x$

8. $\int \sec x \tan x\, dx = \sec x + c$ because $\frac{d}{dx}(\sec x) = \sec x \tan x$

Also

$\frac{d}{dx}(\cot x) = -\csc^2 x$

and $\frac{d}{dx}(\csc x) = -\csc x \cot x$

So, 9) $\int \csc^2 x\, dx = -\cot x + c$

10) $\int \csc x \cot x = -\csc x + c$

Example 7: Find the general antiderivative of $f(x) = \frac{1}{2}$.

$F(x) = \int \frac{1}{2}\, dx = \boxed{\frac{1}{2}x + c}$

check: $\frac{d}{dx}\left(\frac{1}{2}x + c\right) = \frac{1}{2}$ ✓

Power rule:
$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (\text{for } n \neq -1)$$

Example 8: Find $\int x^3\,dx$.

$$\int x^3\,dx = \frac{x^{3+1}}{3+1} + c = \boxed{\frac{x^4}{4} + c}$$

Check: $\frac{d}{dx}\left(\frac{x^4}{4}\right) = \frac{d}{dx}\left(\frac{1}{4}x^4\right) = \frac{1}{\cancel{4}}\cdot\cancel{4}x^3 = x^3$ ✓OK

Example 9: Find $\int 7x^2\,dx$.

$$\int 7x^2\,dx = 7\int x^2\,dx = 7\cdot\frac{x^3}{3} + c = \boxed{\frac{7x^3}{3} + c} = \boxed{\frac{7}{3}x^3 + c}$$

$$\frac{d}{dx}\left(\frac{7}{3}x^3 + c\right) = \frac{7}{3}\cdot 3x^2 = 7x^2 \checkmark$$

$$\frac{d}{dx}\left(\frac{7x^3}{3}\right) = \frac{21x^2}{3} = 7x^2 \checkmark$$

Example 10: Find $\int \frac{1}{x^5}\,dx$.

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{x^{-5+1}}{-5+1} + c$$

$$= \frac{x^{-4}}{-4} + c = \boxed{-\frac{1}{4x^4} + c}$$

check: $\frac{d}{dx}\left(-\frac{1}{4}x^{-4}\right)$

$$= -\frac{1}{4}\left(-4x^{-5}\right) = x^{-5} = \frac{1}{x^5}$$ ✓OK

Example 11: Find the general antiderivative of $f(x) = \frac{5}{x^2}$.

$f(x) = 5x^{-2}$

Antiderivative $F(x) = \int 5x^{-2}\,dx = 5\cdot\frac{x^{-1}}{-1} + c = \boxed{-\frac{5}{x} + c}$

Check $\frac{d}{dx}\left(-5x^{-1}\right)$

$$= 5x^{-2} = \frac{5}{x^2} \checkmark$$

Example 12: $\int\left(6x^2 - 3x + 9\right)dx$

$$\int(6x^2 - 3x + 9)\,dx = 6\cdot\frac{x^3}{3} - 3\cdot\frac{x^2}{2} + 9x + C$$

$$= \boxed{2x^3 - \frac{3x^2}{2} + 9x + C}$$

Check: $\frac{d}{dx}\left(2x^3 - \frac{3}{2}x^2 + 9x\right)$

$$= 6x^2 - \frac{3}{2}\cdot 2x + 9$$

$$= 6x^2 - 3x + 9 \checkmark$$

Example 13: Find $\int 3\sqrt{x}\,dx$.

$$\int 3\sqrt{x}\,dx = 3\int x^{\frac{1}{2}}\,dx$$

$$= 3\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{3x^{3/2}}{3/2} + c = \cancel{3}\cdot\frac{2}{\cancel{3}}x^{3/2} + c$$

$$= \boxed{2x^{3/2} + c}$$

Example 14: Find $\int (3\cos x + 5\sin x)\,dx$.

$$\int (3\cos x + 5\sin x)\,dx = 3\int \cos x\,dx + 5\int \sin x\,dx$$
$$= 3\sin x + C_1 + 5(-\cos x + C_2)$$
$$= 3\sin x - 5\cos x + \underbrace{C C_1 + 5C_2}_{C}$$
$$= \boxed{3\sin x - 5\cos x + C}$$

Example 15: Find $\int \frac{x^7 - \sqrt[3]{x} + 3x^2}{x^5}\,dx$.

$$\int \frac{x^7 - x^{1/3} + 3x^2}{x^5}\,dx = \int x^{-5}(x^7 - x^{1/3} + 3x^2)\,dx$$
$$= \int (x^2 - x^{-5+\frac{1}{3}} + 3x^{-3})\,dx = \int (x^2 - x^{-14/3} + 3x^{-3})\,dx = \frac{x^3}{3} - \frac{x^{-\frac{14}{3}+1}}{-\frac{14}{3}+1} + \frac{3x^{-2}}{-2}$$
$$= \frac{x^3}{3} - \frac{x^{-11/3}}{-11/3} - \frac{3}{2}x^{-2} + C$$
$$= \boxed{\frac{x^3}{3} + \frac{3}{11}x^{-11/3} - \frac{3}{2}x^{-2} + C}$$
$$= \frac{x^3}{3} + \frac{3}{11\sqrt[3]{x^{11}}} - \frac{3}{2x^2} + C$$

Example 16: Find the general antiderivative of $f(\theta) = \frac{\sin\theta}{3}$.

$$F(\theta) = \int \frac{\sin\theta}{3}\,d\theta = \frac{1}{3}\int \sin\theta\,d\theta$$
$$= \frac{1}{3}(-\cos\theta) + C$$
$$= \boxed{-\frac{1}{3}\cos\theta + C}$$

check it!

Example 17: $\int \left(\sqrt[3]{x} + \frac{2}{\sqrt{x}}\right)dx$

$$\int \left(\sqrt[3]{x} + \frac{2}{\sqrt{x}}\right)dx = \int (x^{1/3} + 2x^{-1/2})\,dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + \frac{2x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$
$$= \frac{x^{4/3}}{4/3} + \frac{2x^{1/2}}{1/2} + C = \frac{3}{4}x^{4/3} + \frac{2}{1}\cdot 2x^{1/2} + C$$
$$= \boxed{\frac{3}{4}x^{4/3} + 4x^{1/2} + C}$$

Example 18: $\int (6y^2 - 2)(8y + 5)\,dy$

$$\int (48y^3 + 30y^2 - 16y - 10)\,dy$$
$$= 48\cdot\frac{y^4}{4} + 30\cdot\frac{y^3}{3} - 16\cdot\frac{y^2}{2} - 10y + C$$
$$= \boxed{12y^4 + 10y^3 - 8y^2 - 10y + C}$$

Check: Taking derivative $\Rightarrow 48y^3 + 30y^2 - 16y - 10$ ✓

Differential equations:

A *differential equation* is an equation involving the derivative of a function. To solve a differential equation means to find the original function.

(DE) or ODE

An *initial value problem* is a common type of differential equation in which a derivative and an initial condition are given.

Example 19: Given $f'(x) = x^2 - 7$, find f. This is an example of a differential equation.

We need the antiderivative.

$$f(x) = \int f'(x)\,dx = \int (x^2-7)\,dx = \boxed{\frac{x^3}{3} - 7x + C.}$$

general solution

Need more info to find C.

Example 20: Suppose that $f'(x) = 3x^2 + 2\cos x$ and $f(0) = 3$. Find $f(x)$.

(initial value problem)

(DE with an initial value)

$$f(x) = \int f'(x)\,dx = \int (3x^2 + 2\cos x)\,dx$$

$$= \frac{3x^3}{3} + 2\sin x + C$$

$$= x^3 + 2\sin x + C.$$ Check it!

Use $f(0)=3$ to find C.

$$f(0) = 0^3 + 2\sin(0) + C = 3$$

$$0 + 0 + C = 3$$

$$C = 3$$

Solution of my DE: $\boxed{f(x) = x^3 + 2\sin x + 3}$

Example 21: Suppose that $f''(x)=2x^3-6x^2+6x$, $f'(2)=-1$, and $f(-1)=4$. Find $f(x)$.

$$f'(x)=\int f''(x)\,dx=\int(2x^3-6x^2+6x)\,dx=\frac{2x^4}{4}-\frac{6x^3}{3}+\frac{6x^2}{2}+c_1$$

$$=\frac{x^4}{2}-2x^3+3x^2+c_1$$

$$f'(2)=\frac{2^4}{2}-2(2)^3+3(2)^2+c_1=-1$$

$$8-16+12+c_1=-1$$

$$4+c_1=-1$$

$$c_1=-5$$

$$\boxed{f(x)=\frac{1}{10}x^5-\frac{1}{2}x^4+x^3-5x+\frac{3}{5}}$$

$$f'(x)=\frac{x^4}{2}-2x^3+3x^2-5$$

$$f(x)=\int f'(x)\,dx=\int\left(\frac{x^4}{2}-2x^3+3x^2-5\right)dx=\frac{1}{2}\cdot\frac{x^5}{5}-\frac{2x^4}{4}+3\frac{x^3}{3}-5x+c_2$$

$$=\frac{1}{10}x^5-\frac{1}{2}x^4+x^3-5x+c_2$$

$$f(-1)=\frac{1}{10}(-1)^5-\frac{1}{2}(-1)^4+(-1)^3-5(-1)+c_2=4$$

$$=-\frac{1}{10}-\frac{1}{2}-1+5+c_2=4\Rightarrow-\frac{1}{10}-\frac{5}{10}+\frac{40}{10}+c_2=4$$

$$\frac{34}{10}+c_2=4$$

$$c_2=\frac{40}{10}-\frac{34}{10}=\frac{6}{10}=\frac{3}{5}$$

Example 22: Suppose that $f''(x)=12x^2-18x$, $f(1)=2$, and $f(-3)=1$. Find $f(x)$.

$$f'(x)=\int f''(x)\,dx=\int(12x^2-18x)\,dx=\frac{12x^3}{3}-\frac{18x^2}{2}+c_1=4x^3-9x^2+c_1$$

$$f(x)=\int f'(x)\,dx=\int(4x^3-9x^2+c_1)\,dx=\frac{4x^4}{4}-\frac{9x^3}{3}+c_1x+c_2$$

$$f(x)=x^4-3x^3+c_1x+c_2$$

$$f(1)=2\Rightarrow f(1)=1^4-3(1)^3+c_1(1)+c_2=2$$

$$1-3+c_1+c_2=2$$

$$-2+c_1+c_2=2$$

$$c_1+c_2=4$$

$$f(-3)=1\Rightarrow(-3)^4-3(-3)^3+c_1(-3)+c_2=1$$

$$81+81-3c_1+c_2=1$$

$$162-3c_1+c_2=1$$

$$-3c_1+c_2=-161$$

system of 2 equations in 2 unknowns. Can solve by substitution or elimination. Solve $c_1+c_2=4$ for c_1:

$$c_1=4-c_2$$

Substitute $c_1=4-c_2$ into $-3c_1+c_2=-161$

$$-3(4-c_2)+c_2=-161$$

$$-12+3c_2+c_2=-161$$

$$4c_2=-149$$

$$c_2=-\frac{149}{4}=-37.25$$

$$c_1+c_2=4$$

$$c_1-37.25=4$$

$$c_1=41.25$$

$$\boxed{f(x)=x^4-3x^3+41.25x-37.25}$$

Velocity and acceleration (rectilinear motion):

We already know that if $f(t)$ is the position of an object at time t, then $f'(t)$ is its velocity and $f''(t)$ is its acceleration.

Note: Acceleration due to gravity near the earth's surface is approximately 9.8 m/s^2 or 32 ft/s^2.

Example 23: Suppose a particle's velocity is given by $v(t) = 2\sin t + \cos t$ and its initial position is $s(0) = 3$. Find the position function of the particle.

$v(t) = 2\sin t + \cos t = s'(t)$ where $s(t)$ is the position at time t.

$s(t) = \int v(t)\,dt = \int s'(t)\,dt = \int (2\sin t + \cos t)\,dt = -2\cos t + \sin t + c$

$s(0) = 3 \Rightarrow s(0) = -2\cos(0) + \sin(0) + c = 3$

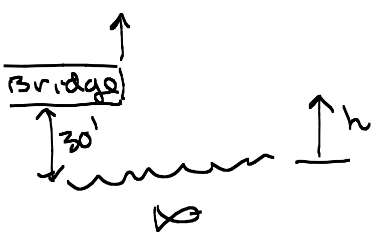
$-2(1) + 0 + c = 3$

$c = 5$

$$\boxed{s(t) = -2\cos t + \sin t + 5}$$

Example 24: Suppose a ball is thrown upward from a 30-foot bridge over a river at an initial velocity of 40 feet per second. How high does it go? When does it hit the water?

Let h represent the distance above the water. (in feet)



acceleration: $a(t) = h''(t) = -32.2$ ft/sec^2

velocity: $v(t) = h'(t) = \int h''(t)\,dt$

$= \int (-32.2)\,dt$

$= -32.2t + c_1$

initial condition: $v(0) = h'(0) = 40 \Rightarrow -32.2(0) + c_1 = 40$

$c_1 = 40$

$v(t) = h'(t) = -32.2t + 40$

position: $h(t) = \int v(t)\,dt = \int h'(t)\,dt = \int (-32.2t + 40)\,dt$

$= -32.2\frac{t^2}{2} + 40t + c_2 = -16.1t^2 + 40t + c_2$

initial condition: $h(0) = 30 \Rightarrow -16.1(0)^2 + 40(0) + c_2 = 30$

$0 + 0 + c_2 = 30$

see next page

Ex 24 cont'd: $C_2 = 30$ 4.1.9

$$\boxed{h(t) = -16.1t^2 + 40t + 30}$$ position function

Note: For acceleration $= -g$ (g = acceleration due to gravity)

$a(t) = h''(t) = \int -g \, dt$

$h'(t) = -gt + c_1$

$v(0) = v_0 \Rightarrow$ $v(t) = h'(t) = -g(0) + c_1 = v_0$

$c_1 = v_0$

$v(t) = h'(t) = -gt + v_0$

$h(t) = \int h'(t)dt = \int(-gt + v_0)dt$

$= -g\frac{t^2}{2} + v_0 t + C_2$

$h(0) = h_0 \Rightarrow$

$-\frac{g}{2}(0)^2 + v_0(0) + C_2 = h_0$

$C_2 = h_0$

g = acceleration due to gravity
v_0 = initial velocity
h_0 = initial height
(positive = up)

$$\boxed{h(t) = -\frac{gt^2}{2} + v_0 t + h_0}$$

How high does it go?
When does it hit the water?

At max height: $s'(t) = 0$ (velocity = 0)

$-32t + 40 = 0$

$40 = 32t$

$t = \frac{40}{32} = \frac{5}{4} = 1.25$ sec

max height is $s(1.25) = -16(1.25)^2 + 40(1.25) + 30 =$ $\boxed{55 \text{ ft max height}}$

see next page

Ex 24 cont'd

When it hits the water, $h(t)=0$:

$$-16t^2 + 40t + 30 = 0$$

Quadratic formula:

$$t = \frac{-40 \pm \sqrt{40^2 - 4(-16)(30}}{2(-16)}$$

$$\approx \frac{20 \pm \sqrt{880}}{16} \approx 3.104 \text{ sec}, -0.604 \text{ sec}$$

We want 3.104 seconds.

$\boxed{\text{It hits the water at } t = 3.104 \text{ seconds.}}$