3.1: Extrema on an Interval

Absolute maximum and minimum:

If $f(x) \le f(c)$ for every x in the domain of f, then $f(c)$ is the *maximum*, or *absolute maximum*, of f.

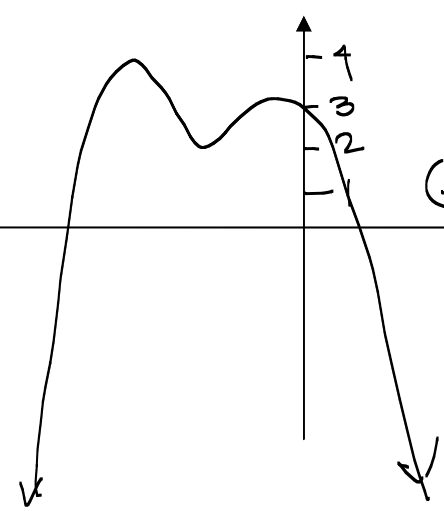
If $f(x) \ge f(c)$ for every x in the domain of f, then $f(c)$ is the *minimum*, or *absolute minimum* of f.

The maximum and minimum values of a function are called the *extreme values* of the function.

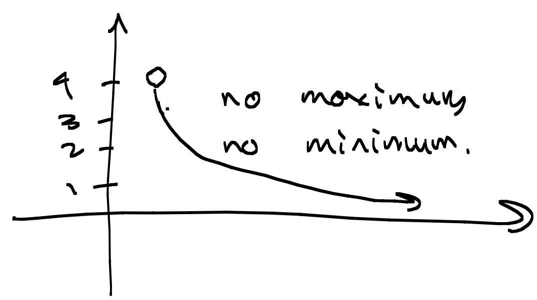
In other words, (global)

- The *absolute maximum* is the largest y-value on the graph.
- The *absolute minimum* is the smallest y-value on the graph.

Example 1:



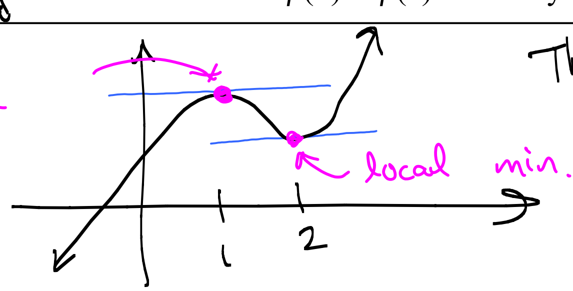
The absolute maximum is 4.
There is no absolute minimum.
(equivalent to: the maximum is 4, there is no minimum.)



Notice: if the function is smooth, the tangent line will be horizontal at the local max/min

Relative (Local) Maxima and Minima: (these are relative extrema)

- A function f has a *relative maximum*, or *local maximum*, at $x = c$ if there is an interval (a,b) around c such that $f(x) \le f(c)$ for every x in (a,b). (These are the "hilltops").
- A function f has a *relative minimum*, or *local minimum*, at $x = c$ if there is an interval (a,b) around c such that $f(x) \ge f(c)$ for every x in (a,b). (These are the "bottoms of valleys").



This function does not have an absolute max or min. It has a local max at 1 and a local min at 2.

(smooth = differentiable)

Notice: If the function is differentiable (smooth), then the tangent line at a local minimum or maximum is horizontal.

> Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

This means that if f is differentiable at c and has a relative extreme at c, then the tangent line to f at c must be horizontal.

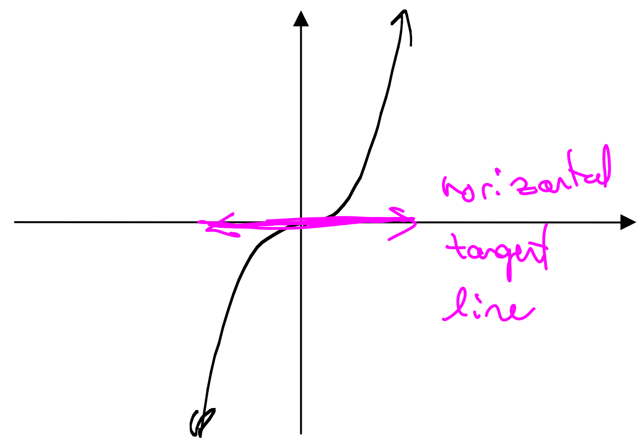
However, we must be careful. The fact that $f'(c) = 0$ (tangent line is horizontal) does not guarantee that there is a relative minimum or maximum at c.

Example 2: $f(x) = x^3$

$f'(x) = 3x^2$

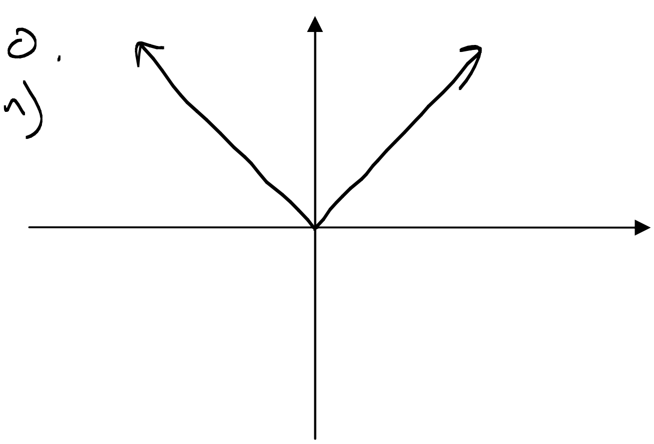
where is $f'(x) = 0$? where $x = 0$.

$f'(0) = 3(0)^2 = 0$



Example 3: There can be a local maximum or minimum at c even if $f'(c)$ does not exist.

has a local min at 0.
(also is the absolut min)
but $f'(0)$ does not exist
(sharp corner)



Critical numbers:

<u>Critical Number</u>: A *critical number* of a function *f* is a number *c* in the domain of *f* such that either $f'(c)=0$ or $f'(c)$ does not exist.

<u>Theorem</u>: If *f* has a local maximum or minimum at *c*, then *c* is a critical number of *f*. <u>Note</u>: The converse of this theorem is not true. It is possible for *f* to have a critical number at *c*, but not to have a local maximum or minimum at *c*.

Example 4: Find the critical numbers of $f(x)=x^3+\frac{17}{2}x^2-6x+4$.

$f'(x)=3x^2+\frac{17}{2}(2x)-6$

$=3x^2+17x-6$

set $f'(x)=0$: $0=3x^2+17x-6$

$0=(3x-1)(x+6)$

$3x-1=0 \quad | \quad x+6=0$

$x=\frac{1}{3} \leftarrow 3x=1 \quad | \quad x=-6$

$\boxed{\text{Critical numbers: } -6 \text{ and } \frac{1}{3}}$

Example 5: Find the critical numbers of $f(x)=x^{2/3}$ defined on $(-\infty,\infty)$

$f(x)=x^{2/3}=\sqrt[3]{x^2}$

$f'(x)=\frac{2}{3}x^{-1/3}=\frac{2}{3\sqrt[3]{x}}$

Where is $f'(x)=0$? $f'(x)$ is never 0 because the numerator cannot be 0.

Where is $f'(x)$ undefined? when $x=0$ So $\boxed{\text{0 is the only critical number.}}$

Example 6: Find the critical numbers of $f(x)=\frac{x^2}{x-3}$ Domain: $\{x \mid x\neq 3\}$

$f'(x)=\frac{(x-3)\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(x-3)}{(x-3)^2}=\frac{(x-3)(2x)-x^2(1)}{(x-3)^2}=\frac{2x^2-6x-x^2}{(x-3)^2}$

$=\frac{x^2-6x}{(x-3)^2}=\frac{x(x-6)}{(x-3)^2}$

set $f'(x)=0$: $0=\frac{x(x-6)}{(x-3)^2}$

$0=x(x-6)$

$x=0,\ x=6$

where is $f'(x)=0$: at 0 and 6.

Where is $f'(x)$ undefined? when $x=3$

Is 3 in the domain of f? No

So 3 is not a critical number.

$\boxed{\text{0 and 6 are critical numbers}}$

Absolute extrema on a closed interval:

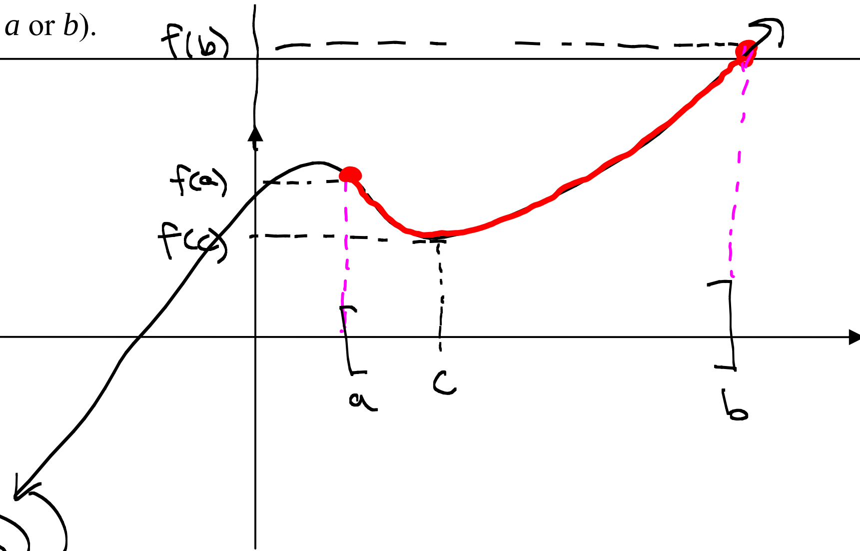
Extreme Value Theorem: If f is continuous on a closed interval $[a,b]$, then f has both an absolute maximum and an absolute minimum on $[a,b]$.

Note: The absolute maximum and the absolute minimum must occur at either a critical value in (a,b) or at an endpoint (at a or b).

Example 7:

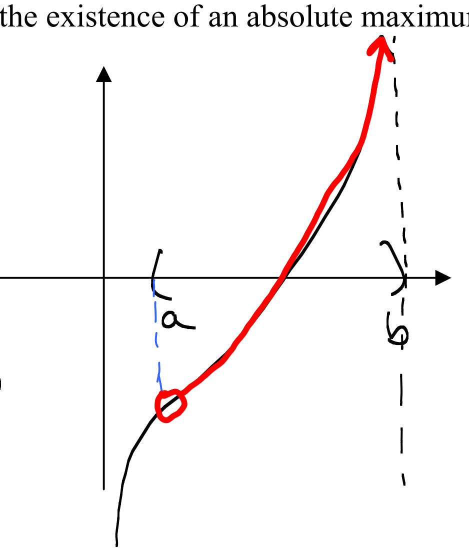
Here, the absolute max is $f(b)$ (an endpoint)

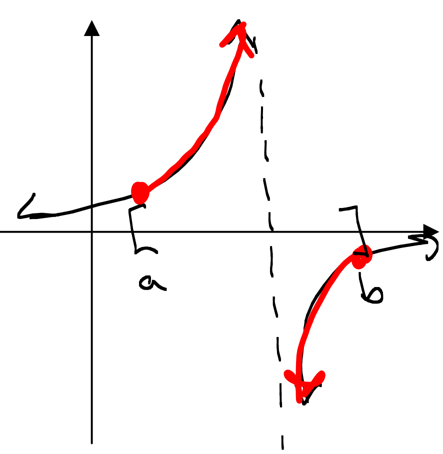
The absolute min is $f(c)$ (at a critical number in the interior (a,b))



Example 8: If either hypothesis of the extreme value theorem (continuity or closed interval) is violated, the existence of an absolute maximum or minimum is not guaranteed.

No absolute max or abs min on (a,b)





Process for Finding the Absolute Extrema of a Continuous Function on a Closed Interval:

1. Find the critical values in (a,b).
2. Compute the value of f at each critical value in (a,b) and also compute $f(a)$ and $f(b)$.
3. The absolute maximum is the largest of these y-values and the absolute minimum is the smallest of these y-values.

Example 9: Find the absolute extrema for $f(x) = x^2 + 2$ on the interval $[-2,3]$.

Find critical values: $f'(x) = 2x$

set $f'(x) = 0$: $2x = 0$

$x = 0$ is this in $[-2,3]$? yes

Candidates: $f(-2) = (-2)^2 + 2 = 6$

$f(0) = 0^2 + 2 = 2$ ← smallest

$f(3) = 3^2 + 2 = 11$ ← largest

The absolute maximum is $f(3) = 11$.
The absolute minimum is $f(0) = 2$

Example 10: Find the extreme values of $g(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 + 3$ on the interval $[-2,1]$.

g is continuous, domain $(-\infty, \infty)$

$g'(x) = \frac{1}{2}(4x^3) - \frac{2}{3}(3x^2) - 4x$

$= 2x^3 - 2x^2 - 4x = 2x(x^2 - x - 2)$

$= 2x(x-2)(x+1)$

critical numbers: $0, 2, -1$
(from setting $f'(x) = 0$)

only 0 an -1 are in interval $[-2,1]$. So ignore 2 ($2 \notin [-2,1]$)

critical #s

$g(0) = \frac{1}{2}(0)^4 - \frac{2}{3}(0)^3 - 2(0)^2 + 3 = 3$

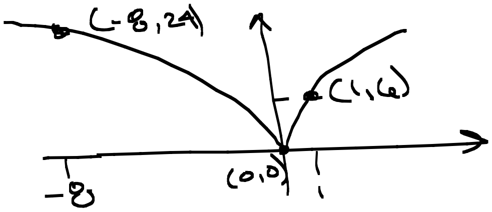
$g(-1) = \frac{1}{2}(-1)^4 - \frac{2}{3}(-1)^3 - 2(-1)^2 + 3 = \frac{1}{2} + \frac{2}{3} - 2 + 3 = \frac{3}{6} + \frac{4}{6} + \frac{6}{6} = \frac{13}{6} = 2\frac{1}{6}$

endpts

$g(-2) = \frac{1}{2}(-2)^4 - \frac{2}{3}(-2)^3 - 2(-2)^2 + 3 = 8 + \frac{16}{3} - 8 + 3 = \frac{25}{3} = 8\frac{1}{3}$

$g(1) = \frac{1}{2}(1)^4 - \frac{2}{3}(1)^3 - 2(1)^2 + 3 = \frac{1}{2} - \frac{2}{3} - 2 + 3 = \frac{3}{6} - \frac{4}{6} + \frac{6}{6} = \frac{5}{6}$

Absolute max: $g(-2) = 8\frac{1}{3}$
Absolute min: $g(1) = \frac{5}{6}$



Example 11: Find the absolute extrema of $h(x) = 6x^{2/3}$ on the intervals (a) $[-8,1]$, (b) $[-8,1)$, and. (c) $(-8,1)$.

$h(x) = 6x^{2/3} = 6\sqrt[3]{x^2}$. Domain is $(-\infty, \infty)$.

h is continuous on its domain.

$h'(x) = 6\left(\frac{2}{3}x^{-\frac{1}{3}}\right) = 4x^{-1/3} = \frac{4}{\sqrt[3]{x}}$

h' is undefined for $x=0$.
0 is in domain of h, so 0 is a critical number.

h' is never 0 (because numerator is never 0).

0 is the only critical number.

Find y-values.

$h(0) = 6\sqrt[3]{(0)^2} = 0$

$h(-8) = 6\sqrt[3]{(-8)^2} = 6\sqrt[3]{64} = 6 \cdot 4 = 24$

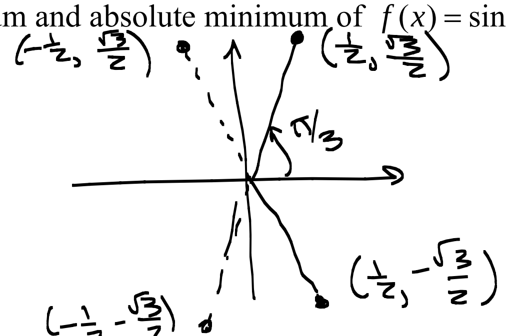
$h(1) = 6\sqrt[3]{(1)^2} = 6 \cdot 1 = 6$

(a) On $[-8,-1]$, absolute minimum is $h(0) = 0$, and absolute maximum is $h(-8) = 24$.

(b) on $[-8,1)$, absolute max is still $h(-8) = 24$ and abs. min is $h(0) = 0$.

(c) on $(-8,1)$, the abs. min is $h(0) = 0$. No absolute max.

Example 12: Find the absolute maximum and absolute minimum of $f(x) = \sin 2x - x$ on the interval $[0, \pi]$.



f is continuous on $(-\infty, \infty)$.

$f'(x) = \cos(2x)(2) - 1$
$= 2\cos 2x - 1$

Set $f'(x) = 0$: $0 = 2\cos(2x) - 1$
$1 = 2\cos(2x)$
$\frac{1}{2} = \cos(2x)$

$0 \le x \le \pi$ Given
$0 \le 2x \le 2\pi$

$2x = \frac{\pi}{3}$, $2x = \frac{5\pi}{3}$
$x = \frac{\pi}{6}$ $x = \frac{5\pi}{6}$ both in $[0, \pi]$

$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \approx 0.34247$

$f\left(\frac{5\pi}{6}\right) = \sin\left(\frac{10\pi}{6}\right) - \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6} \approx -3.48402$

$f(0) = \sin(2 \cdot 0) - 0 = \sin 0 - 0 = 0$

$f(\pi) = \sin(2\pi) - \pi = 0 - \pi = -\pi \approx -3.14159$

$$\boxed{\begin{array}{l}\text{Absolute maximum is } f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}. \\ \text{Absolute minimum is } f\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6}.\end{array}}$$