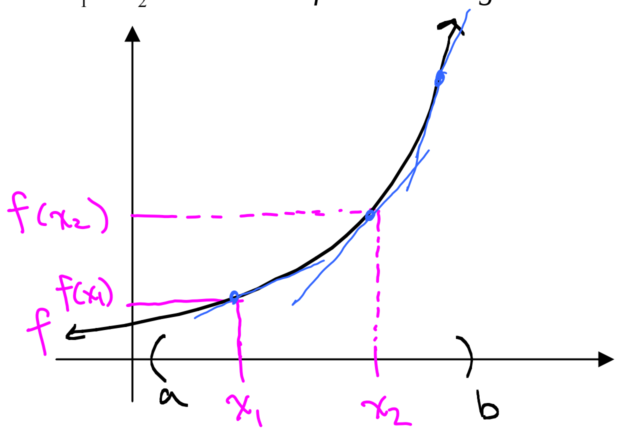
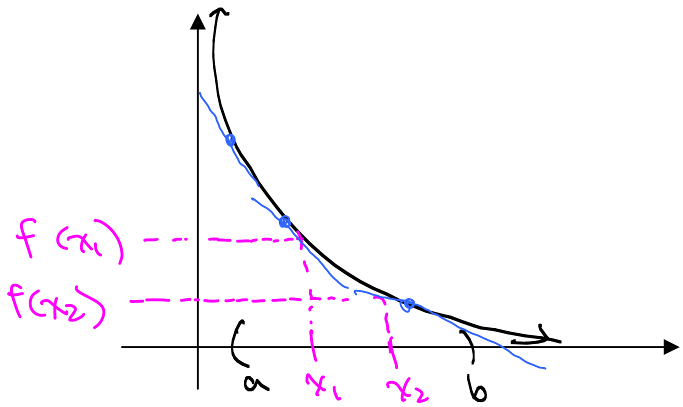
3.3: Increasing and Decreasing Functions and the First Derivative Test

Increasing and decreasing functions:

A function *f* is said to be *increasing* on the interval (a,b) if, for any two numbers x_1 and x_2 in (a,b), $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. A function *f* is *increasing at c* if there is an interval around *c* on which *f* is increasing.



A function *f* is said to be *decreasing* on the interval (a,b) if, for any two numbers x_1 and x_2 in (a,b), $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. A function *f* is *decreasing at c* if there is an interval around *c* on which *f* is decreasing.



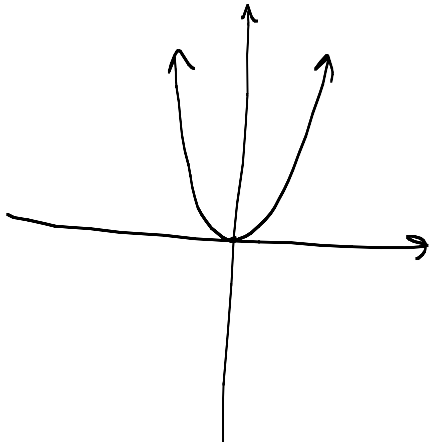
Notice that wherever a function is increasing, the tangent lines have positive slope.
Notice that wherever a function is decreasing, the tangent lines have negative slope.

This means that we can use the derivative to determine the intervals where a function is increasing and decreasing.

<u>Increasing/Decreasing Test</u>: Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

- If $f'(x) > 0$ for every *x* in (a,b), then *f* is <u>increasing</u> on (a,b).
- If $f'(x) < 0$ for every *x* in (a,b), then *f* is <u>decreasing</u> on (a,b).
- If $f'(x) = 0$ for every *x* in (a,b), then *f* is <u>constant</u> on (a,b).

Example 1: $f(x) = x^2$



f is decreasing on $(-\infty, 0)$.
f is increasing on $(0, \infty)$.

$f'(x) = 2x$

For $x < 0$, $f'(x) = 2x < 0$

For $x > 0$, $f'(x) = 2x > 0$

Steps for Determining Increasing/Decreasing Intervals 1. Find all the values of x where $f'(x) = 0$ or where $f'(x)$ is not defined. Use these values to split the number line into intervals. (Find domain and critical numbers) 2. Choose a test number c in each interval and determine the sign of $f'(c)$. • If $f'(c) > 0$, then f is increasing on that interval. • If $f'(c) < 0$, then f is decreasing on that interval.

Note: Three types of numbers can appear on your number line:
1) Numbers where the function is defined and the derivative is 0. (These are critical numbers.)
2) Numbers where the function is defined and the derivative is undefined. (These are also critical numbers.)
3) Numbers where the function is undefined. (These are NOT critical numbers.)

First derivative test:

This procedure determines the relative extrema of a function *f*.

First derivative test: Suppose that c is a critical number of a function f that is continuous on an open interval containing c. rel max • If $f'(x)$ changes from positive to negative across c, then f has a relative maximum at c. • If $f'(x)$ changes from negative to positive across c, then f has a relative minimum at c. rel - min • If $f'(x)$ does not change sign across c, then f does not have a relative extreme at c.

Example 2: Determine the intervals on which $f(x) = x^3 + 6x^2 - 36x + 18$ is increasing and decreasing. Find the relative extrema.

Domain: $(-\infty, \infty)$

$$f'(x) = 3x^2 + 12x - 36$$
$$= 3(x^2 + 4x - 12)$$
$$= 3(x+6)(x-2)$$

$f'(x) = 0 \Rightarrow x = -6,\ x = 2$

Critical values: $-6, 2$

f': + + + + | − − − − − | + + + +

increasing -6 decreasing 2 increasing

$(-\infty, -6)$: Test number $x = -7$

$$f'(-7) = 3(-7)^2 + 12(-7) - 36$$
$$= 3(49) - 84 - 36$$
$$= 147 - 84 - 36$$
$$= 27 \quad (+)$$

so $f'(x) > 0$ on $(-\infty, -6)$.

Interval $(-6, 2)$: Test $x = 0$

$$f'(0) = 3(0)^2 + 12(0) - 36 = -36 \quad (-)$$

So $f'(x) < 0$ on $(-6, 2)$

Interval $(2, \infty)$: Test number $x = 3$

$$f'(3) = 3(3)^2 + 12(3) - 36$$
$$= 27 + 36 - 36$$
$$= 27 \quad (+)$$

so $f'(x) > 0$ on $(2, \infty)$

$\boxed{f \text{ is increasing on } (-\infty, -6) \text{ and on } (2, \infty)\quad f \text{ is decreasing on } (-6, 2).}$

Relative max at $x = -6$
relative min at $x = 2$.

Find the y-values:

$$f(-6) = (-6)^3 + 6(-6)^2 - 36(-6) + 18 = 234$$
$$f(2) = 2^3 + 6(2)^2 - 36(2) + 18 = -22$$

$\boxed{\text{Relative max is } f(-6) = 234;\ \text{rel. min is } f(2) = -22}$

Example 3: Determine the intervals on which $g(x) = x^3 - 6x^2 + 12x - 8$ is increasing and decreasing. Find the relative extrema.

Domain: $(-\infty, \infty)$

$$g'(x) = 3x^2 - 12x + 12$$
$$= 3(x^2 - 4x + 4)$$
$$= 3(x-2)^2$$

$g'(x) = 0 \Rightarrow x = 2$

critical number: 2

+ + + + + + + | + + + + +

increasing 2 increasing

$(-\infty, 2)$: Test $x = 0$

$$g'(x) = 3(x-2)^2$$
$$g'(0) = 3(0-2)^2$$
$$\Rightarrow (+)(-)^2$$
$$\Rightarrow (+)$$

$(2, \infty)$: Test $x = 3$

$$g'(3) = 3(3-2)^2$$
$$= 3(1)^2 = 3 \quad (+)$$
$$3(+)^2 \Rightarrow (+)$$

$\boxed{g \text{ is increasing on } (-\infty, \infty).}$

$\boxed{\text{No relative extrema.}}$

Example 4: Determine the intervals on which $g(x) = x^{2/5}$ is increasing and decreasing. Find the relative extrema.

$g(x) = x^{2/5} = \sqrt[5]{x^2}$

Domain: $(-\infty, \infty)$

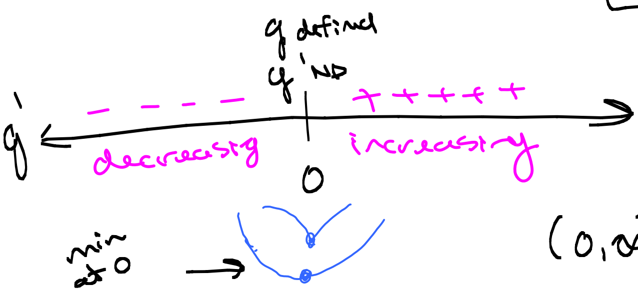
$g'(x) = \frac{2}{5} x^{-3/5} = \frac{2}{5\sqrt[5]{x^3}}$

g is decreasing on $(-\infty, 0)$.
g is increasing on $(0, \infty)$

Where is $g'(x) = 0$? never (numerator is never 0)
Where is $g'(x)$ undefined? At $x = 0$.
So $\boxed{0 \text{ is the only critical value.}}$

Relative minimum is $g(0) = 0$.

$g(0) = \sqrt[5]{0^2} = 0$



$(-\infty, 0)$: Test $x = -1$

$g'(-1) = \frac{2}{5\sqrt[5]{(-1)^3}} = \frac{2}{5\sqrt[5]{-1}} = \frac{2}{5(-1)} = -\frac{2}{5}$ $(-)$

$(0, \infty)$: Test $x = 1$

$g'(1) = \frac{2}{5\sqrt[5]{(1)^3}} = \frac{2}{5\sqrt[5]{1}} = \frac{2}{5(1)} = +\frac{2}{5}$

Example 5: Determine the intervals on which $f(x) = x + \frac{4}{x}$ is increasing and decreasing. Find the relative extrema.

$f(x) = x + 4x^{-1}$ Domain: $x \neq 0$

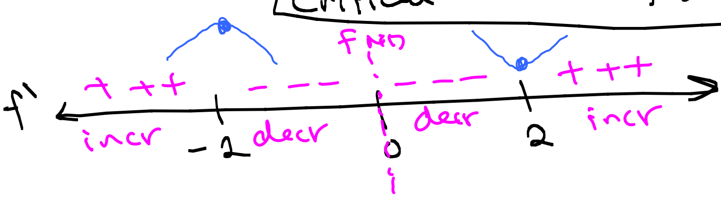
$f(x) = x + \frac{4}{x} = \frac{x^2 + 4}{x}$

$f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2} = \frac{x^2}{x^2} - \frac{4}{x^2}$

$= \frac{x^2 - 4}{x^2} = \frac{(x+2)(x-2)}{x^2}$

where is $f'(x) = 0$? At $x = \pm 2$ (where numerator $= 0$)
where is $f'(x)$ undefined? at $x = 0$ (where denominator $= 0$)
is f defined at $x = 0$? No. So 0 is not a critical number.

$\boxed{\text{Critical values: } 2, -2}$



$(-\infty, -2)$: Test $x = -3$

$f'(x) = \frac{(x+2)(x-2)}{x^2}$

$f'(-3) = \frac{(-3+2)(-3-2)}{(-3)^2} \Rightarrow \frac{(-)(-)}{(+)} \Rightarrow (+)$

$(2, \infty)$ Test: $x = 3$

$f'(3) = \frac{(3+2)(3-2)}{(3)^2} \Rightarrow \frac{(+)(+)}{(+)} \Rightarrow (+)$

$(-2, 0)$: Test $x = -1$

$f'(-1) = \frac{(-1+2)(-1-2)}{(-1)^2} \Rightarrow \frac{(+)(-)}{(+)} \Rightarrow (-)$

$(0, 2)$: Test $x = 1$

$f'(1) = \frac{(1+2)(1-2)}{(1)^2} \Rightarrow \frac{(+)(-)}{(+)} \Rightarrow (-)$

Increasing on $(-\infty, -2)$ and $(2, \infty)$
Decreasing on $(-2, 0)$ and $(0, 2)$
Relative max: $f(-2) = -4$
Relative min: $f(2) = 4$

Example 6: Find the local extremes of $g(x) = (x^2 - 4)^{2/3}$. Where is it increasing and decreasing?

See Archive Notes for Summer 2014

Example 7: Find the relative extremes of $f(x) = \frac{1}{2}x - \sin x$ on the interval $(0, 2\pi)$. Where is it increasing and decreasing on that interval?

See archive notes Summer 2014.