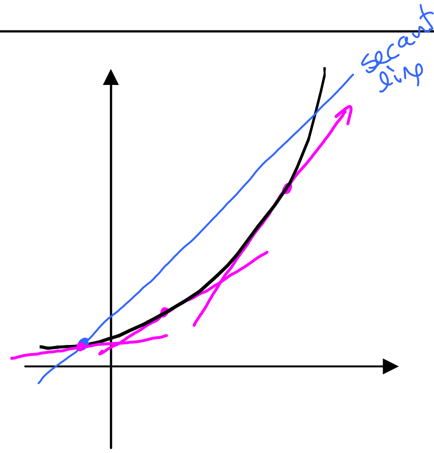
3.4: Concavity and the Second Derivative Test

Concavity:

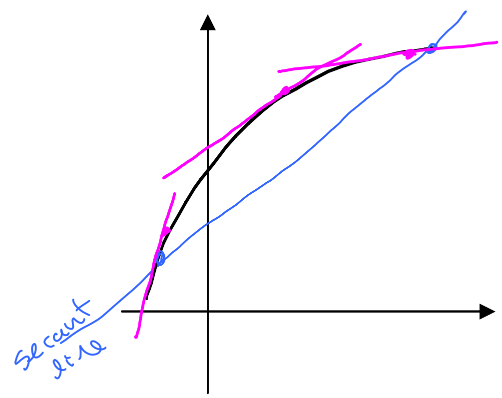
Definition:

- If the graph of f lies above all of its tangents on an interval, then it is called *concave upward* on that interval. (below the secant line)
- If the graph of f lies below all its tangents on an interval, it is called *concave downward* on that interval. (above the secant lines)

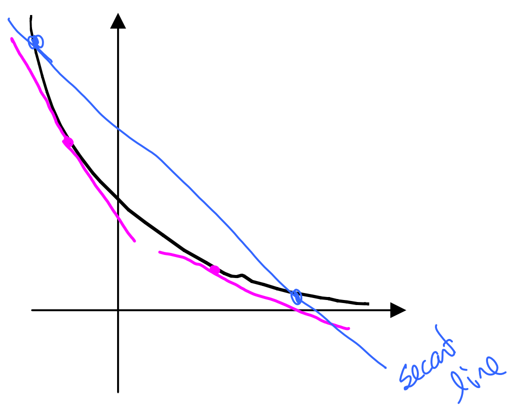
Illustration:



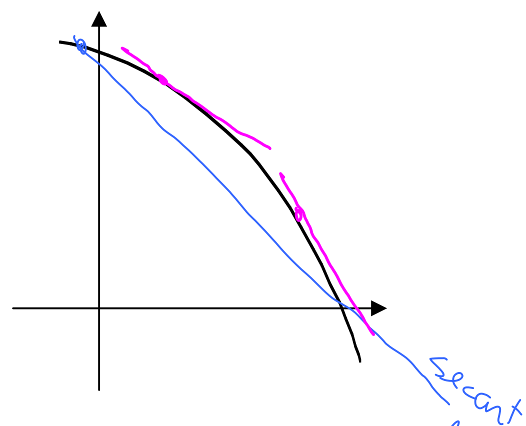
Increasing and Concave Up



Increasing and Concave Down



Decreasing and Concave Up



Decreasing and Concave Down

Notice the slopes of the tangent lines. When the curve is concave up, the slopes are increasing as you move from left to right.

When the curve is concave down, the slopes are decreasing as you move from left to right.

We find out whether f' is increasing or decreasing by looking at its derivative, which is f''.

Concavity Test:

- If $f''(x) > 0$ for all x in (a,b), then f is concave up on (a,b).
- If $f''(x) < 0$ for all x in (a,b), then f is concave down on (a,b).

Process for Determining Intervals of Concavity:

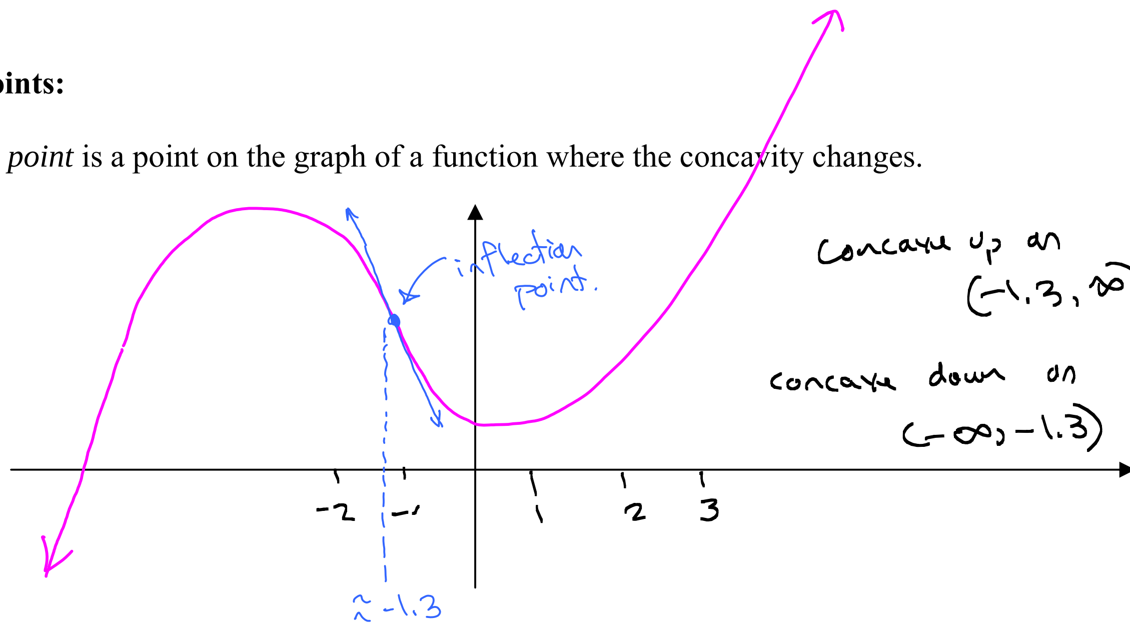
1. Find the values of x where $f''(x) = 0$ or where $f''(x)$ is not defined. Use these values of x to divide the number line into intervals.

2. Choose a test number c in each interval.
   - If $f''(c) > 0$, then f is concave up on that interval.
   - If $f''(c) < 0$, then f is concave down on that interval.

Inflection points:

An *inflection point* is a point on the graph of a function where the concavity changes.

Example 1:



concave up on $(-1.3, \infty)$

concave down on $(-\infty, -1.3)$

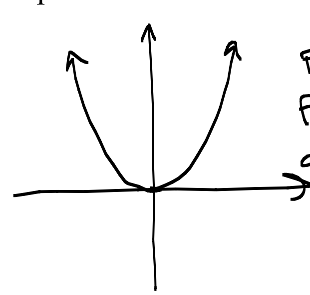
Example 2: Find the intervals on which $f(x) = x^2$ is concave up and concave down.

$f'(x) = 2x$

$f''(x) = 2$

positive for every x

concave up on $(-\infty, \infty)$



From graph, $f(x) = x^2$ is concave up on $(-\infty, \infty)$

Example 3: Determine the intervals of concavity and the inflection points of $f(x) = x^3 + 6x^2 - 36x + 18$.

Domain: $(-\infty, \infty)$

$f'(x) = 3x^2 + 12x - 36$

$f''(x) = 6x + 12$

set $f''(x) = 0$: $0 = 6x + 12$

$0 = 6(x+2)$

$x = -2$

f'': $- - - - - - - - \mid + + + + + + + +$

concave down, -2, concave up

$(-\infty, -2)$: Test number $x = -3$

$f''(x) = 6(x+2)$

$f''(-3) = 6(-3+2) = -6$ $(-)$

$(-2, \infty)$: Test number $x = -1$

$f''(-1) = 6(-1+2) = 6(1) = 6$ $(+)$

or $f''(-1) = 6(-1) + 12 = 6$

$\boxed{\text{concave up on } (-2, \infty) \quad \text{concave down on } (-\infty, -2)}$

Inflection point at $x = -2$

Find y-value:

$f(-2) = (-2)^3 + 6(-2)^2 - 36(-2) + 18$

$= 106$

$\boxed{\text{Inflection point: } (-2, 106)}$

Example 4: Determine the intervals of concavity and the inflection points of $f(x) = x + \frac{4}{x}$.

Domain: $x \neq 0$

$(-\infty, 0) \cup (0, \infty)$

$f(x) = x + 4x^{-1}$

$f'(x) = 1 - 4x^{-2}$

$f''(x) = 8x^{-3} = \frac{8}{x^3}$

Where is $f''(x) = 0$? Never. (numerator is never 0)

Where is $f''(x)$ undefined? At $x = 0$

Is the original function defined at 0? No

f'': $- - - - - - \mid + + + + + + +$

f ND

concave down, 0, concave up

$(-\infty, 0)$: Test $x = -1$

$f''(-1) = \frac{8}{(-1)^3} = -8$ $(-)$

$(0, \infty)$: Test $x = 1$

$f''(1) = \frac{8}{1^3} = 8$ $(+)$

$\boxed{\text{Concave down on } (-\infty, 0). \quad \text{Concave up on } (0, \infty). \quad \text{No inflection points.}}$

No inflection point at $x = 0$ because 0 is not in the domain of f.

The second derivative test:

Notice: For a smooth (differentiable) function, the graph is concave upward at a relative minimum and concave downward at a relative maximum.

Therefore, at a critical number, we can look at the sign of f'' to determine whether there is a relative minimum or relative maximum at that critical number.

> The Second Derivative Test (for Local Extremes):
>
> Suppose f'' is continuous near c. (so f' and f must be defined)
> - If $f'(x) = 0$ and $f''(c) < 0$, then f has a relative maximum at c.
> - If $f'(x) = 0$ and $f''(c) > 0$, then f has a relative minimum at c.
> - If $f'(x) = 0$ and $f''(c) = 0$, then the test is inconclusive. Use the 1st derivative test instead.

Example 5: Use the second derivative test to find the local extremes of $f(x) = x^3 + 6x^2 - 36x + 18$.

$f'(x) = 3x^2 + 12x - 36$

$f''(x) = 6x + 12$

Find critical values: Set $f'(x) = 0$: $3x^2 + 12x - 36 = 0$

$3(x^2 + 4x - 12) = 0$

$3(x-2)(x+6) = 0$

$x = 2, -6$ critical values

Put the critical values into the 2nd derivative:

$f''(2) = 6(2) + 12 = 24$ (+)

relative min at $x = 2$

concave up

$f''(-6) = 6(-6) + 12 = -36 + 12 = -24$ (−)

rel max

concave down

relative max at $x = -6$

Find the y-values:

$f(2) = 2^3 + 6(2)^2 - 36(2) + 18$
$= -22$

$f(-6) = (-6)^3 + 6(-6)^2 - 36(-6) + 18$
$= 234$

Relative min: $f(2) = -22$

Relative max $f(-6) = 234$

See next page

Example 6: Determine the local extremes of $f(x) = -2x^4 + 4x^3$

$f'(x) = -8x^3 + 12x^2$
$= -4x^2(2x - 3)$

$f''(x) = -24x^2 + 24x$
$= -24x(x - 1)$

Find critical values:
set $f'(x) = 0$: $x = 0$, $x = \frac{3}{2}$

$0 = -4x^2(2x - 3)$

$0 = -4x^2$ | $2x - 3 = 0$

$2x = 3$

$x = 3/2$

Evaluate f'' at the critical values:

$x = 0$: $f''(0) = -24(0)(0 - 1)$
$= 0$

2nd derivative test is inconclusive at 0
must use 1st derivative test

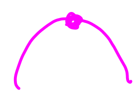
Ex 6 cont'd: $f''(x) = -24x(x-1)$

$\boxed{x = \frac{3}{2}}$ $f''(\frac{3}{2}) = f''(1.5) = -24(1.5)(1.5-1)$

$\Rightarrow (-)(+)(+)$

$\Rightarrow (-)$ concave down

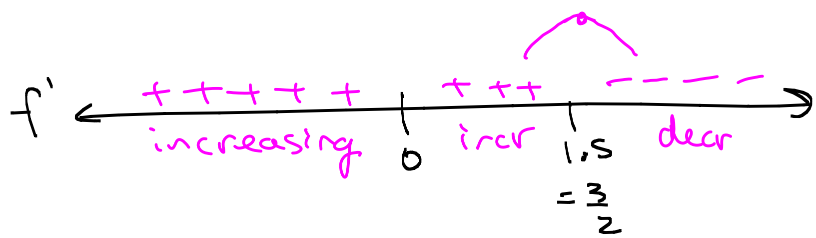
relative max at $x = \frac{3}{2}$



1st derivative Test:

$f'(x) = -8x^3 + 12x^2$

$= -4x^2(2x-3)$



$(-\infty, 0)$: Test $x = -1$

$f'(-1) = -8(-1)^3 + 12(-1)^2$

$= -8(-1) + 12(1)$

$= 8 + 12 = 20$

$(+)$

$(0, 1.5)$: Test $x = 1$

$f'(1) = -8(1)^3 + 12(1)^2$

$= -8 + 12 = 4$

$(+)$

$(1.5, \infty)$: Test $x = 2$

$f'(2) = -8(2)^3 + 12(2)^2$

$= -64 + 48$

$(-)$

No sign change in f' across 0, so no relative min/max at 0.

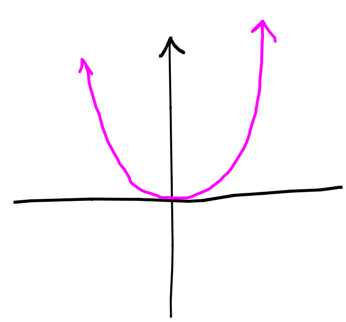
$\boxed{\text{Relative max at } x = \frac{3}{2}}$

(should find y-value)

Ex: $f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$



critical value is $x = 0$.

$f''(0) = 0$ yet we have a relative min at 0.