3.6: A Summary of Curve Sketching

Steps for Curve Sketching

1. Determine the domain of *f*.
2. Find the *x*-intercepts and *y*-intercept, if any.
3. Determine the "end behavior" of *f*, that is, the behavior for large values of $|x|$ (limits at infinity).
4. Find the vertical, horizontal, and oblique asymptotes, if any.
5. Determine the intervals where *f* is increasing/decreasing.
6. Find the relative extremes of *f*, if any. (You should find both the *x*- and *y*-values.)
7. Determine the intervals where *f* is concave up/concave down.
8. Find the inflection points, if any. (You should find both the *x*- and *y*-values.)
9. Plot more points if necessary, and sketch the graph.

Example 1: Sketch the graph of $f(x) = x^3 - 6x^2 + 9x$.

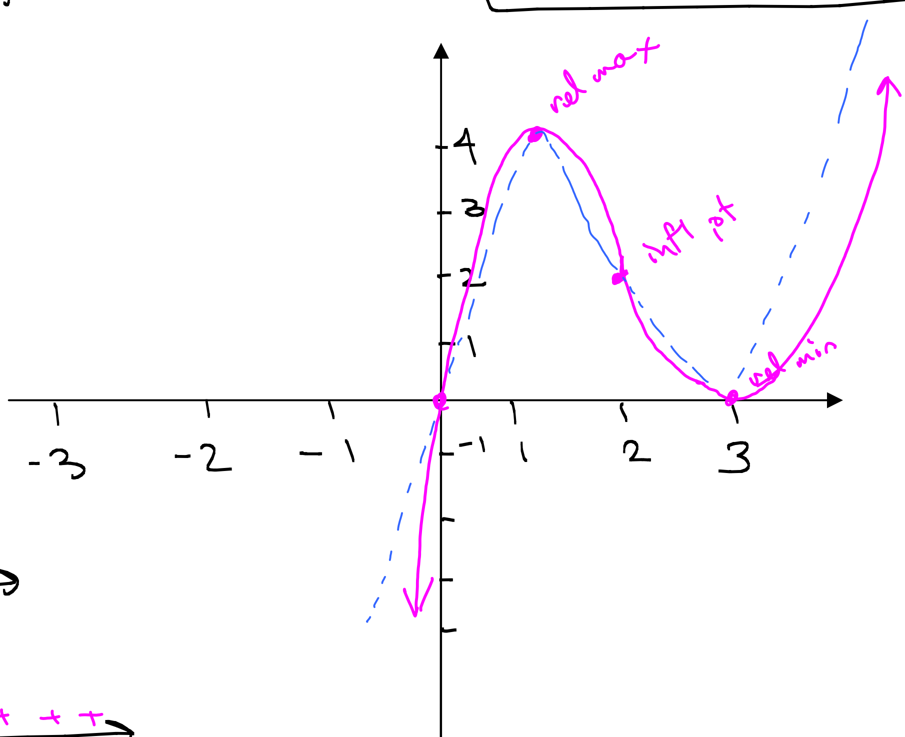
Find x-intercepts: Set $y=0$:

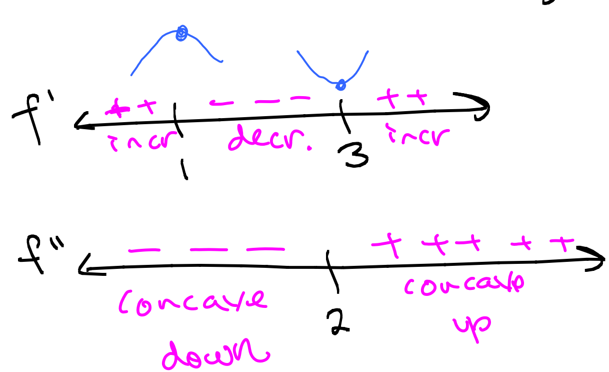
$$0 = x^3 - 6x^2 + 9x$$
$$0 = x(x^2 - 6x + 9)$$
$$0 = x(x-3)^2$$
$$x = 0, 3$$

Find y-intercept: Set $x=0$:

$$f(0) = 0^3 - 6(0)^2 + 9(0) = 0$$

Domain: $(-\infty, \infty)$
x-intercepts: $(0,0)$, $(3,0)$
y-intercept: $(0,0)$
relative max: $(1,4)$
relative min $(3,0)$
inflection pt: $(2,2)$





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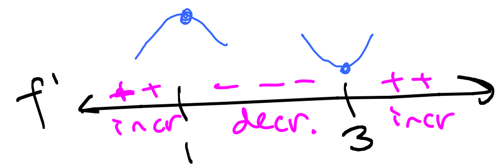
Ex 1 cont'd: $f(x) = x^3 - 6x^2 + 9x$

$f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$



Find critical #s: $f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x-3)(x-1)$

$x = 1, 3$ critical values

$(-\infty, 1)$: Test $x = 0$

$f'(0) = 3(0)^2 - 12(0) + 9 = 9$
$(+)$

$(1, 3)$: Test $x = 2$

$f'(2) = 3(2-3)(2-1)$
$\Rightarrow (+)(-)(+) \Rightarrow (-)$

$(3, \infty)$: Test $x = 4$:

$f'(4) = 3(4-3)(4-1)$
$\Rightarrow (+)(+)(+)$
$\Rightarrow (+)$

Rel. max at $x = 1$, rel. min at $x = 3$.

Find y-values: $f(x) = x^3 - 6x^2 + 9x$

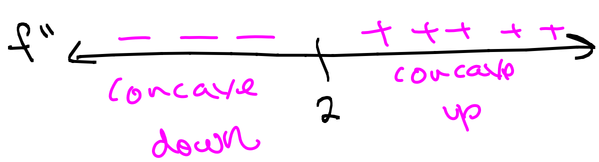
$f(1) = 1^3 - 6(1)^2 + 9$
$= 1 - 6 + 9 = 4$

Rel max at $(1, 4)$.

$f(3) = 3^3 - 6(3)^2 + 9(3) = 27 - 54 + 27$
$= 0$

Rel. min at $(3, 0)$.

Find concavity: $f''(x) = 6x - 12$
$= 6(x-2)$



Set $f''(x) = 0$: $0 = 6(x-2)$
$x = 2$

$(-\infty, 2)$: Test $x = 0$

$f''(0) = 6(0) - 12 = -12$
$(-)$

$(2, \infty)$: Test $x = 3$

$f''(3) = 6(3) - 12$
$(+)$

Inflection point at $x = 2$.

Find the y-value:

$f(2) = 2^3 - 6(2)^2 + 9(2)$

$= 8 - 24 + 18$

$= 2$

Inflection Point: $(2, 2)$

Find y-value for rel. min:
$f(-1) = 3(-1)^4 + 4(-1)^3 = 3 - 4 = -1$

Domain: $(-\infty, \infty)$

Example 2: Sketch the graph of $f(x) = 3x^4 + 4x^3$.

Find x-intercepts: set $y=0$: $f(x) = x^3(3x+4)$
$0 = x^3(3x+4)$
$x = 0 \mid 3x+4=0$
$3x = -4$
$x = -4/3 = -1\frac{1}{3}$

Find y-intercept: Set $x=0$: $f(0) = 3(0)^4 + 4(0)^3 = 0$

Find increasing/decreasing intervals:
$f'(x) = 12x^3 + 12x^2$
$= 12x^2(x+1)$
Setting $f'(x) = 0$ gives critical #s $0, -1$

$(-\infty, -1)$: Test $x=-2$
$f'(-2) = 12(-2)^2(-2+1)$
$\Rightarrow (+)(+)(-)$
$\Rightarrow (-)$

$(-1, 0)$: Test $x = -0.5$
$f'(-0.5) = 12(-0.5)^2(-0.5+1)$
$\Rightarrow (+)(+)(+)$
$\Rightarrow (+)$

$(0, \infty)$: Large $x \Rightarrow f'(x) > 0$

Concavity intervals:
$(-\infty, -2/3)$: Test $x=-1$
$f''(-1) = 12(-1)(3(-1)+2)$
$\Rightarrow (+)(-)(-3+2)$
$\Rightarrow (+)(-)(-)$
$\Rightarrow (+)$

$(-\frac{2}{3}, 0)$: Test $x = -\frac{1}{3}$
$f''(-\frac{1}{3}) = 12(-\frac{1}{3})(3(-\frac{1}{3})+2)$
$\Rightarrow -4(-1+2)$
$\Rightarrow -4(1) = -4$
$(-)$

$(0, \infty)$: Test $x=1$
$f''(1) = 12(1)(3(1)+2)$
$\Rightarrow (+)(+)(+)$
$\Rightarrow (+)$

x-intercepts: $(0,0)$, $(-\frac{4}{3}, 0)$

y-intercept: $(0,0)$

No asymptotes.

Inflection Pts: $(0,0)$, $(-\frac{2}{3}, -\frac{16}{27})$

Rel. min: $(-1, -1)$

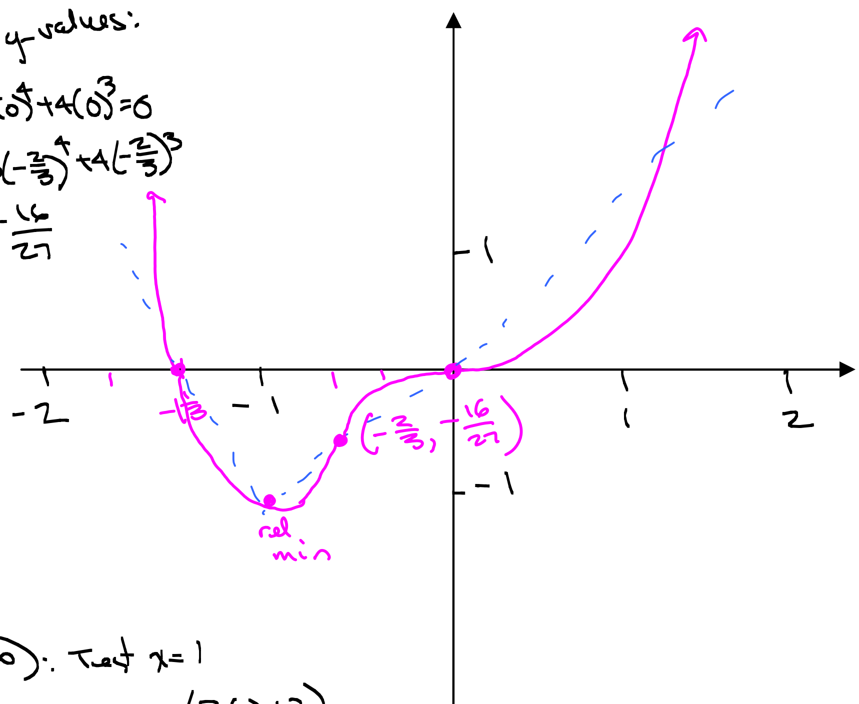
f': $- - -$ decr | -1 | $+ + +$ incr | 0 | $+ + + +$ incr

f'': $+ + + +$ conc up | $-2/3$ | $- -$ conc down | 0 | $+ + +$ concave up

Find concavity
$f''(x) = 36x^2 + 24x$
$= 12x(3x+2)$
Setting $f''(x) = 0 \Rightarrow x = 0$, $3x+2=0$
$3x = -2$
$x = -2/3$

Inflection pts at $x = -2/3$, $x = 0$

Find y-values:
$f(0) = 3(0)^4 + 4(0)^3 = 0$
$f(-2/3) = 3(-\frac{2}{3})^4 + 4(-\frac{2}{3})^3$
$= -\frac{16}{27}$



Domain: $x \neq 1, x \neq -1$

Example 3: Sketch the graph of $f(x) = \dfrac{2x}{x^2-1}$.

$$f'(x) = \frac{-2(x^2+1)}{(x^2-1)^2}$$

$$f''(x) = \frac{4x(x^2+3)}{(x^2-1)^3}$$

Find horizontal asymptote:

$$\lim_{x\to \pm\infty} \frac{2x}{x^2-1} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1-\frac{1}{x^2}} = \frac{0}{1-0} = 0$$

horizontal asymptote: $y=0$

From 1st derivative:

$$f'(x) = \frac{-2(x^2+1)}{(x^2-1)^2} = \frac{-2(x^2+1)}{((x+1)(x-1))^2}$$

where is $f'(x)=0$? Nowhere.
The numerator is never 0.

where is $f'(x)$ undefined? At $x=1, x=-1$

$\boxed{\text{No critical numbers}}$ 1 and −1 are undefined in the original function.

$$f'(x) = \frac{-2(x^2+1)}{(x^2-1)^2}$$

$$\Rightarrow \frac{(-)(+)}{(+)} \Rightarrow (-)$$

$\boxed{\text{Decreasing on } (-\infty,-1) \text{ on } (-1,1) \text{ and } (1,\infty)}$

From 2nd derivative

$$f''(x) = \frac{4x(x^2+3)}{(x^2-1)^3}$$

where is $f''(x)=0$? at $x=0$

Where is $f''(x)$ undefined? At $x=\pm 1$ (same places f is undefined)

$$f(x) = \frac{2x}{(x+1)(x-1)}$$

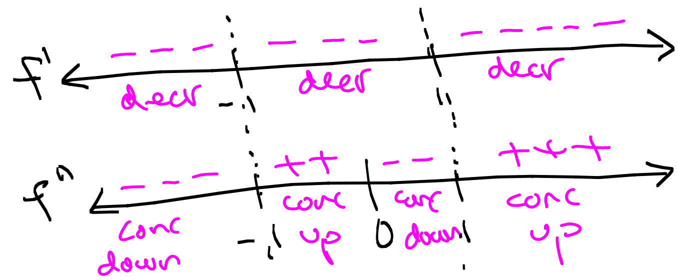
x-intercept: $x=0$ (from setting numerator equal to 0)
$(0,0)$

vertical asymptotes: $x=-1$, $x=1$

y-intercept: 0 or $(0,0)$

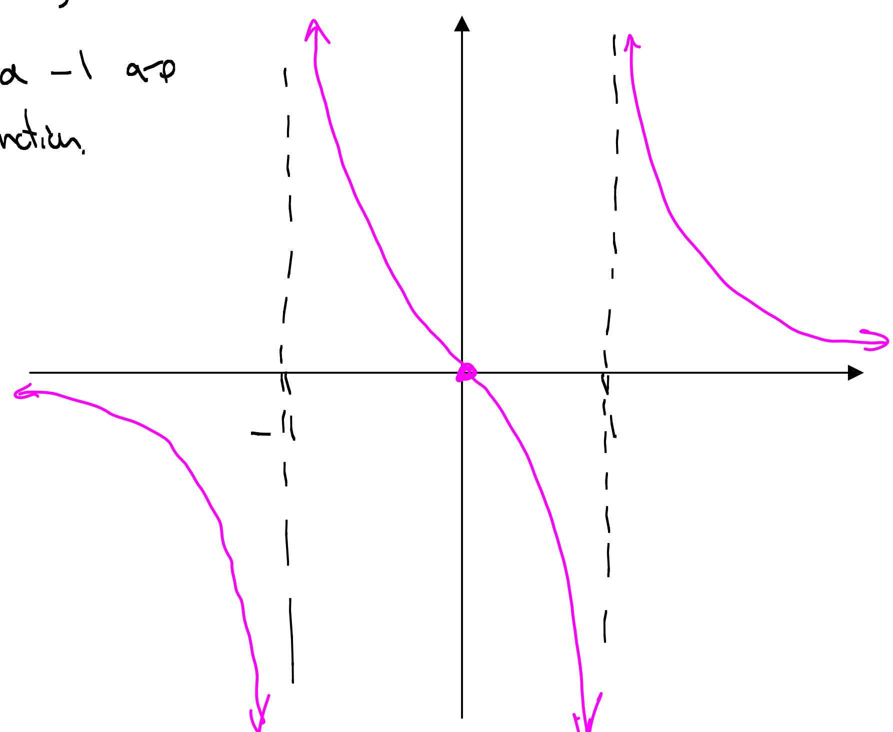
$$f(0) = \frac{2(0)}{0^2-1} = \frac{0}{-1} = 0$$

horiz. asymp: $y=0$



No relative extremas

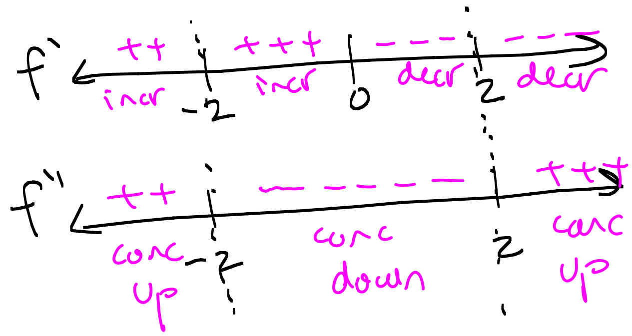
Inflection point: $(0,0)$



Example 4: Sketch the graph of $f(x)=\dfrac{x^2+1}{x^2-4}$.

$$f'(x) = \frac{-10x}{(x^2-4)^2}$$

$$f''(x) = \frac{10(3x^2+4)}{(x^2-4)^3}$$



Vertical asymptotes: $x=2$, $x=-2$

x-intercept: none

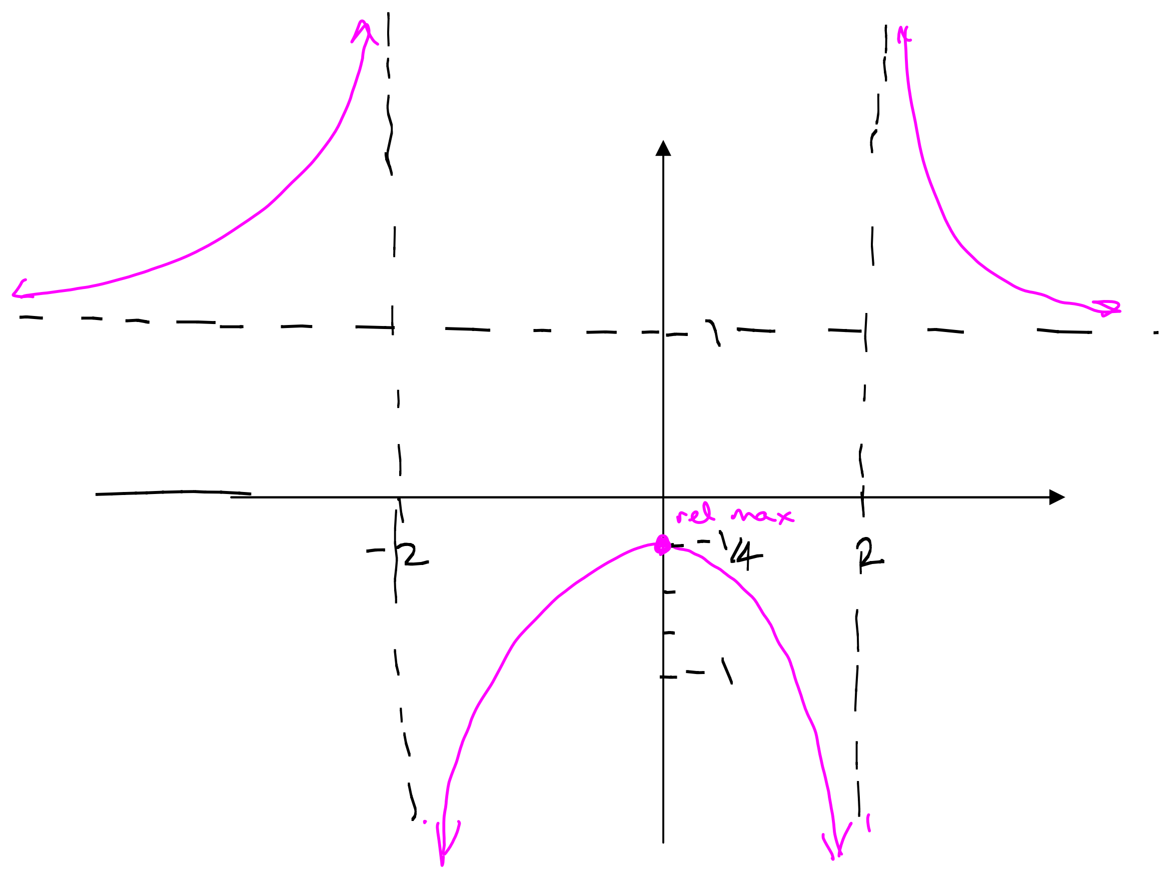
y-intercept: $-\frac{1}{4}$

horizontal asymptote: $y=\frac{1}{1}=1$

Critical values: 0

No inflection points

Relative max: $\left(0, -\frac{1}{4}\right)$



Example 5: Sketch the graph of $f(x) = \dfrac{x^2-4}{x+3}$.

Domain: $x \neq -3$

x-intercepts: $2, -2$

y-intercept: $-\frac{4}{3}$

Vertical asymptote: $x = -3$

No horiz. asymptote

Find horiz. asymp:

$$\lim_{x\to\pm\infty} \frac{x^2-4}{x+3} = \lim_{x\to\pm\infty} \frac{x-\frac{4}{x}}{1+\frac{3}{x}}$$

$$\lim_{x\to+\infty} f(x) = +\infty$$

$$\lim_{x\to-\infty} f(x) = -\infty$$

Find slant asymptote:

$$\begin{array}{r|l} & x \quad -3 \\ \hline x+3 & x^2 + 0x - 4 \\ & -(x^2 + 3x) \\ & \quad -3x - 4 \\ & \quad -(-3x - 9) \\ \hline & \qquad\quad 5 \end{array}$$

$$f(x) = \frac{x^2-4}{x+3} = x - 3 + \frac{5}{x+3}$$

Slant asymptote: $y = x - 3$

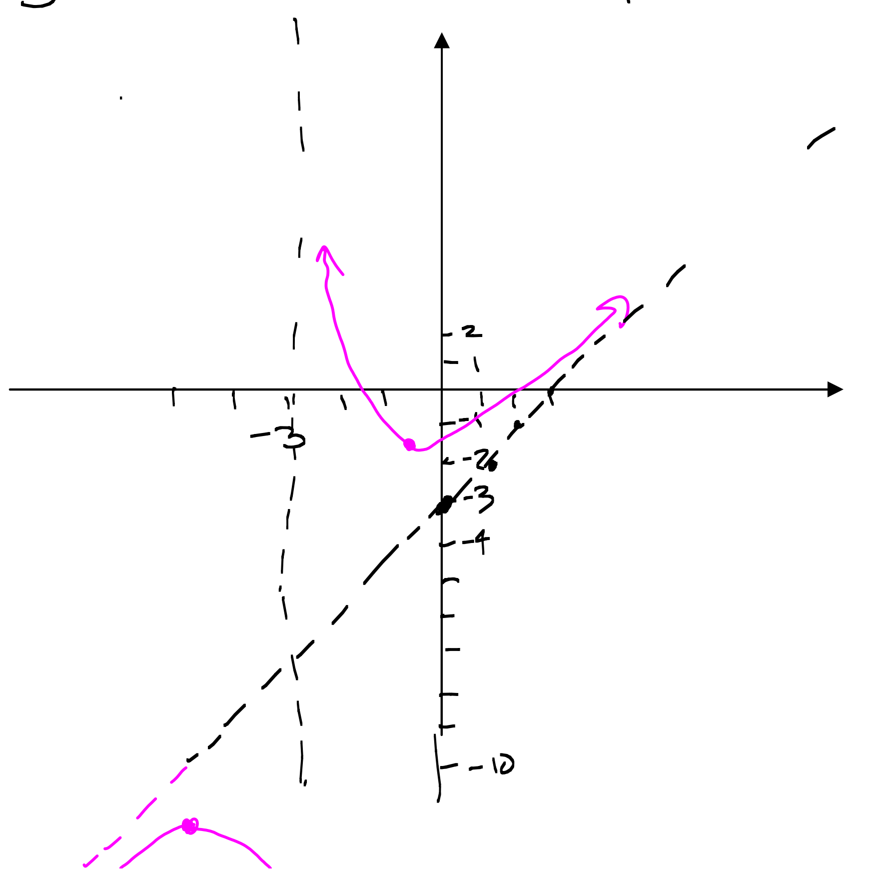
slope 1, y-int -3

f': $+++$ (inc) -5.24 $---$ (decr) -3 $---$ (decr) -0.763 $+++$ (incr)

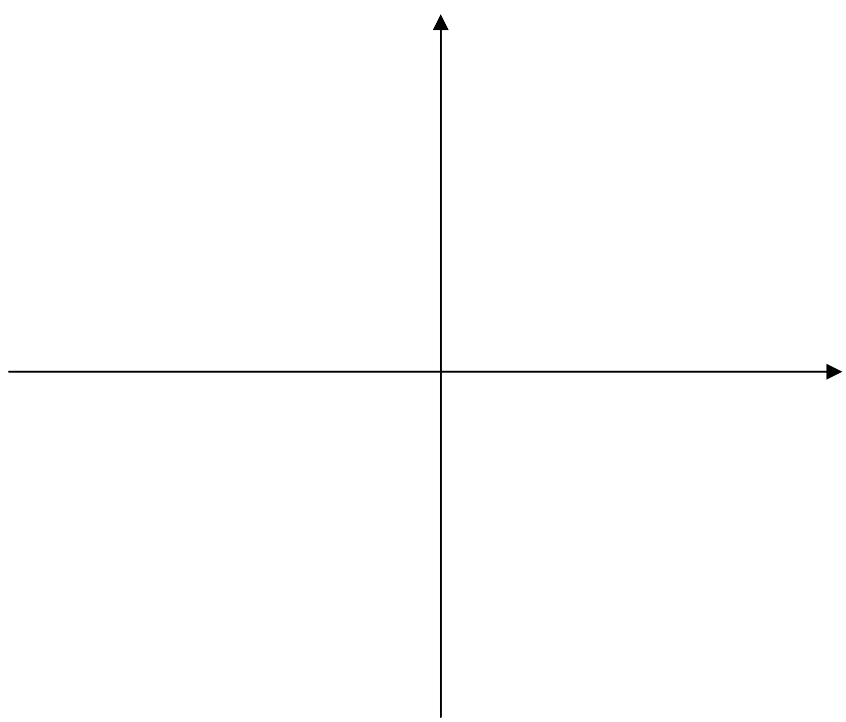
Relative min:

$f(-0.763) \approx -1.5$

Rel max. $f(-5.24) \approx -10$



Example 6: Sketch the graph of $f(x) = 5x^{2/3} - x^{5/3}$.



Example 7: Sketch the graph of $f(x) = x + \cos x$ on the interval $[-2\pi, 2\pi]$.