3.7: Optimization Problems

We often need to solve problems involving optimization: finding the maximum or minimum of some quantity.

Process for Solving Optimization Problems

1. Assign a variable to each quantity mentioned. If possible, draw and label a diagram.
2. Write an expression for the quantity to be optimized.
3. Write the quantity to be optimized as a function of one variable. Determine its domain.
4. Find the minimum or maximum by sketching the curve and finding the relative extrema, or by calculating the absolute maximum or minimum on a closed interval.

Example 1: Find two positive numbers such that the sum of the first and twice the second is 100, and their product is a maximum.

Maximize: Product = P

write an equation for P: $P = xy$

1st number: x

2nd number: y

write an eqn relating x and y to each other: $x + 2y = 100$

solve for x: $x = 100 - 2y$

Put $x = 100 - 2y$ into $P = xy$:

$P = (100 - 2y)(y)$

$P(y) = 100y - 2y^2$

$P'(y) = 100 - 4y$

set $P'(y) = 0$: $100 - 4y = 0$

$100 = 4y$

$25 = y$

2nd derivative test: $P''(y) = -4$ local max

It's a quadratic − parabola opening down, so this must be the absolute max.

Ex 1 cont'd:

Find x:

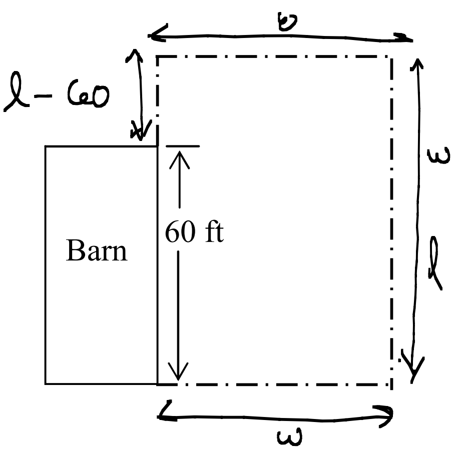
$x = 100 - 2y$

$y = 25 \Rightarrow$

$x = 100 - 2(25) = 50$

The two numbers are 25 and 50.

Example 2: A farmer wants to construct a rectangular pen next to a barn 60 feet long, using the entirety of one side of the barn as part of one side of the pen. Find the dimensions of the pen with the largest area that the farmer can build if 250 feet of fencing material is available.



Maximize: Area = A

write eqn for A: $A = lw$

want to get rid of either l or w, so write an eqn relating l and w.

$2w + l + l - 60 = 250$

$2w + 2l - 60 = 250$

$2w + 2l = 310$

solve for l: $2l = 310 - 2w$

$l = \frac{310}{2} - \frac{2w}{2} = 155 - w$

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Ex 2 cont'd: $A = lw$

substitute $l = 155 - w$: $A = (155-w)(w)$

$A(w) = 155w - w^2$

$A'(w) = 155 - 2w$

Set it equal to 0: $0 = 155 - 2w$

$2w = 155$

$w = \frac{155}{2} = 77.5$

critical value

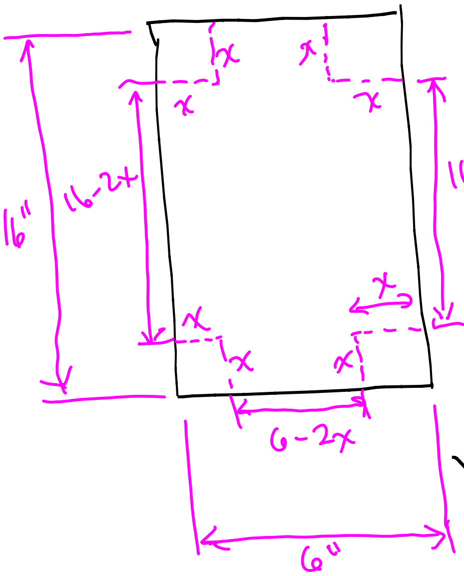
$A(w) = 155w - w^2$ is a quadratic, so a parabola opening down so the absolute max must be at $w = 77.5$

$l = 155 - w$

$w = 77.5 \Rightarrow l = 155 - 77.5 = 77.5$

Dimensions of pen with maximum area are 77.5 ft by 77.5 ft.

Example 3: A rectangular piece of cardboard can be turned into an open box by cutting away squares from the corners and turning up the flaps. If a piece of cardboard is 6 inches wide and 16 inches long, find the dimensions of the box with maximum volume.



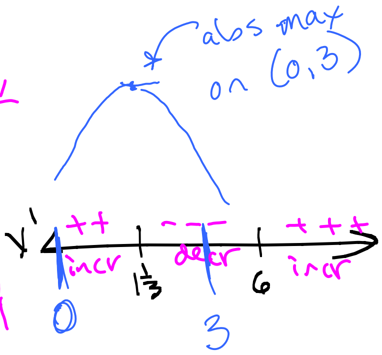
Maximize: Volume = V

write eqn for V: $V = (\text{length})(\text{width})(\text{height})$

$= (16-2x)(6-2x)(x)$

$V(x) = x(96 - 44x + 4x^2)$

$= 96x - 44x^2 + 4x^3$



$V'(x) = 96 - 88x + 12x^2$

$= 4(24 - 22x + 3x^2)$

$= 4(3x^2 - 22x + 24)$

$= 4(3x - 4)(x - 6)$

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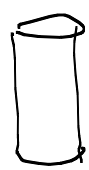
Domain of V: $0 < x < 3$

could think of $[0,3]$ and note that $x=0$ and $x=3$ don't really give us a box.

Set $V'(x) = 0$: $3x - 4 = 0$, $3x = 4$, $x = 4/3 = 1\frac{1}{3}$ | $x - 6 = 0$, $x = 6$ impossible

Example 4: A dog food company decides to package its new dog treats, *Dusty's Yummy Doggy Kibbles*, in cylindrical cans. Each can will be filled to the top with 54 cubic inches of delicious dog treats. What height and radius should be used to minimize the amount of metal required?

Minimize: Surface Area = S



$S = (\text{lateral surface area}) + (\text{area of top}) + (\text{area of bottom})$

$S = 2\pi r h + \pi r^2 + \pi r^2$

need a relationship between r and h

Volume: $54 = \pi r^2 h$

solve for h: $h = \frac{54}{\pi r^2}$

$S = 2\pi r h + 2\pi r^2$

$= 2\pi r\left(\frac{54}{\pi r^2}\right) + 2\pi r^2$

$= \frac{108}{r} + 2\pi r^2$

$S(r) = 108r^{-1} + 2\pi r^2$

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$S'(r) = -108r^{-2} + 4\pi r$

$= -\frac{108}{r^2} + 4\pi r$

$= -\frac{108}{r^2} + \frac{4\pi r}{1}\left(\frac{r^2}{r^2}\right)$

$= \frac{-108 + 4\pi r^3}{r^2}$

$S'(r)$ is undefined at 0.

Domain $r > 0$

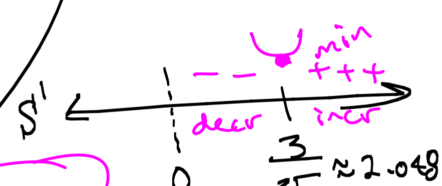
Where is $S'(r) = 0$? wherever $0 = -108 + 4\pi r^3$

$108 = 4\pi r^3$

$\frac{108}{4\pi} = r^3$

$r^3 = \frac{27}{\pi}$

$r = \sqrt[3]{\frac{27}{\pi}} = \frac{3}{\sqrt[3]{\pi}} \approx 2.048$ in



Ex: 4 cont'd:

Find h: $h = \frac{54}{\pi r^2} = \frac{54}{\pi\left(\frac{3}{\sqrt[3]{\pi}}\right)^2}$

$= \frac{54}{\frac{9\pi}{\pi^{2/3}}} = \frac{54}{9\pi^{1/3}} = \frac{6}{\sqrt[3]{\pi}} \approx 4.100$

$\boxed{\text{For minimal surface area, radius is } \frac{3}{\sqrt[3]{\pi}} \approx 2.048 \text{ in and height is } \frac{6}{\sqrt[3]{\pi}} \approx 4.100 \text{ in.}}$

Ex 3 cont'd:

absolute max on $[0,3]$ or $(0,3)$ is at

$x = \frac{4}{3}$.

other dimensions are $16-2x$ and $6-2x$, so

$16-2\left(\frac{4}{3}\right)$
$= 16 - \frac{8}{3} = \frac{48}{3} - \frac{8}{3}$
$= \frac{40}{3} = 13\frac{1}{3}$

$6-2\left(\frac{4}{3}\right)$
$= 6 - \frac{8}{3}$
$= \frac{18}{3} - \frac{8}{3} = \frac{10}{3}$
$= 3\frac{1}{3}$

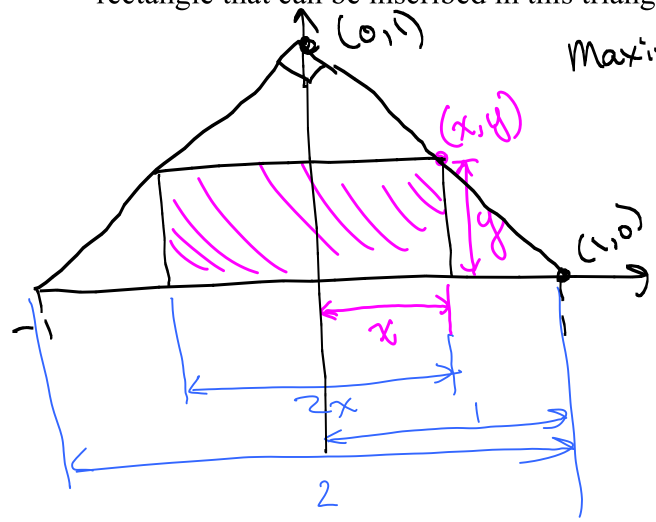
$\boxed{\text{Dimensions of maximum volume box are } 1\tfrac{1}{3}'' \times 13\tfrac{1}{3}'' \times 3\tfrac{1}{3}''.}$

maximum volume is

$\frac{4}{3}\left(\frac{40}{3}\right)\left(\frac{10}{3}\right) = \frac{1600}{27}$

$\approx \boxed{59.26 \text{ in}^3}$

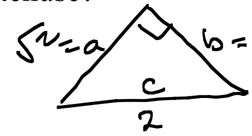
Example 5: An isosceles right triangle has hypotenuse of length 2. What is the maximum area of a rectangle that can be inscribed in this triangle, if one side lies along the hypotenuse?



Maximize: Area = A

$A = (2x)(y)$

$A = 2xy$



$a = b$ because isosceles

$a^2 + b^2 = c^2$

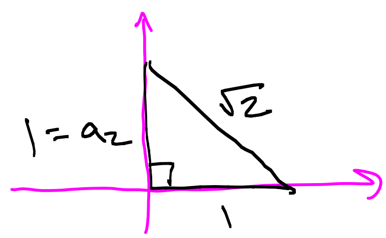
$a^2 + a^2 = c^2$

$2a^2 = c^2$

$a^2 = \frac{c^2}{2}$

$c = 2 \Rightarrow a^2 = \frac{2^2}{2} = \frac{4}{2} = 2$

$a = \sqrt{2}$



$a_2^2 + 1^2 = (\sqrt{2})^2$

$a_2^2 + 1 = 2$

$a_2^2 = 1$

$a_2 = 1$

Eqn of line joining (0,1) and (1,0)

$m = \frac{0-1}{1-0} = -1$

y-intercept: $b = 1$

Eqn of line: $y = -x + 1$

Substitute $y = -x+1$ into $A = 2xy$

$A = 2x(-x+1) = -2x^2 + 2x$

$A'(x) = -4x + 2$

Set $A'(x) = 0$: $-4x + 2 = 0$

$-4x = -2$

$x = \frac{-2}{-4} = \frac{1}{2}$

quadratic opening down, so it's the max.

$y = -x+1$

$x = \frac{1}{2} \Rightarrow y = -\frac{1}{2} + 1 = \frac{1}{2}$

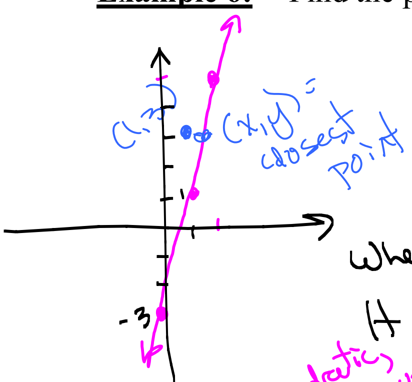
height of rectangle: $y = \frac{1}{2}$

width of rectangle: $2x = 2(\frac{1}{2}) = 1$

$\boxed{\text{Dimensions: } \frac{1}{2}\text{ unit} \times 1\text{ unit}}$

(with 1-unit side along hypotenuse of triangle)

Example 6: Find the point on the line $y = 4x - 3$ that is closest to the point (1, 3).



Minimize: distance D

distance formula: $D = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

When D is small, D^2 will be small also.

It is easier if we minimize D^2 instead of D

Let $Z = D^2 = (x_2-x_1)^2 + (y_2-y_1)^2$

$Z = (x-1)^2 + (y-3)^2$

Z is quadratic, parabola opening up, so $x = \frac{25}{17}$ must be the location of absolute max

write Z as a function of 1 variable by substituting $y = 4x-3$:

$Z = (x-1)^2 + (4x-3-3)^2$

$= (x-1)^2 + (4x-6)^2$

$= x^2 - 2x + 1 + 16x^2 - 48x + 36$

$= 17x^2 - 50x + 37$

$Z'(x) = 34x - 50$

Set $Z'(x) = 0$: $34x - 50 = 0$

$34x = 50$

$x = \frac{50}{34} = \frac{25}{17}$

$y = 4x - 3 = 4\left(\frac{25}{17}\right) - 3 = \frac{100}{17} - 3$

$= \frac{100}{17} - \frac{51}{17} = \frac{49}{17}$

$\boxed{\text{The closest point is } \left(\frac{25}{17}, \frac{49}{17}\right).}$

Example 7: A closed rectangular box is to have a square base and a volume of 20 cubic feet. The material for the base costs 30 cents per square foot, and the material for the sides costs 10 cents per square foot, and the material for the top costs 20 cents per square foot. Determine the dimensions of the box which minimize cost. What is that minimum cost?

See archived notes
for Fall 2014