4.4: The Fundamental Theorem of Calculus

Evaluating the area under a curve by calculating the areas of rectangles, adding them up, and letting taking the limit as $n \to \infty$ is okay in theory but is tedious at best and not very practical.

Fortunately, there is a theorem that makes calculating the area under the curve (definite integral) much easier.

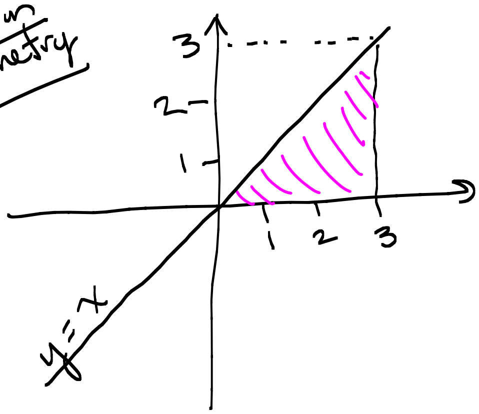
> The Fundamental Theorem of Calculus:
>
> Let f be continuous on the interval $[a,b]$. Then,
>
> $$\int_a^b f(x)dx = F(b) - F(a)$$
>
> where F is any antiderivative of f; in other words, where $F'(x) = f(x)$.

Notation: We'll use this notation when evaluating definite integrals.

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example 1: Find the area under the graph of $f(x) = x$ between 0 and 3.

From geometry



Area $= \int_0^3 x\,dx = \frac{1}{2}(3)(3) = \boxed{\frac{9}{2}}$
$= 4\frac{1}{2}$.

From the Fun Thm of Calculus

$$\int_0^3 x\,dx = \frac{x^2}{2} + C \Big|_0^3$$

$$= \underbrace{\left(\frac{3^2}{2} + C\right)}_{F(b)} - \underbrace{\left(\frac{0^2}{2} + C\right)}_{F(a)}$$

$$= \frac{9}{2} + C - 0 - C$$

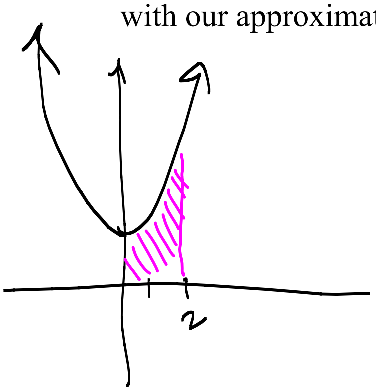
$$= \boxed{\frac{9}{2}}$$

Notice that the constant C disappeared when we evaluated the definite integral. This will always happen.

$$\int_a^b f(x)dx = F(x)+c\Big|_a^b = \big(F(b)+c\big)-\big(F(a)+c\big) = F(b)+c-F(a)-c = F(b)-F(a)$$

So from now on, we'll omit the "$+c$" when evaluating definite integrals.

Example 2: Find the area under the graph of $f(x)=4x^2+1$ over the interval $[0,2]$. (Compare with our approximation in Section 4.2, Example 5).



$$\text{Area} = \int_0^2 (4x^2+1)\,dx = \left(\frac{4x^3}{3}+x\right)\Big|_0^2$$

$$= \frac{4(2)^3}{3}+2-\left(\frac{4(0)^3}{3}+0\right)$$

$$= \frac{32}{3}+2 = \frac{32}{3}+\frac{6}{3} = \frac{38}{3} = \boxed{12\tfrac{2}{3}}$$

Example 3: Evaluate $\int_{-2}^{4}(3x^2-x+4)dx$.

See archive note for remaining examples

Example 4: Evaluate $\int_0^{\pi} (4x^3 + \cos x)dx$.

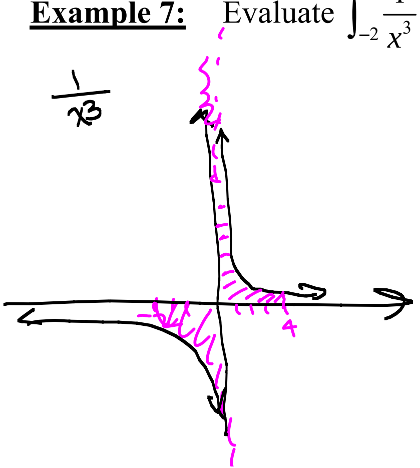
Example 5: Evaluate $\int_1^3 \left(\frac{3}{t^2}\right) dt$.

Example 6: Evaluate $\int_2^9 \frac{1}{\sqrt{u}} du$.

Not an improper integral because $0 \notin [2, 9]$

Example 7: Evaluate $\int_{-2}^4 \frac{1}{x^3} dx$

$f(x) = \frac{1}{x^3}$



Domain: $x \neq 0$
$(-\infty, 0) \cup (0, \infty)$

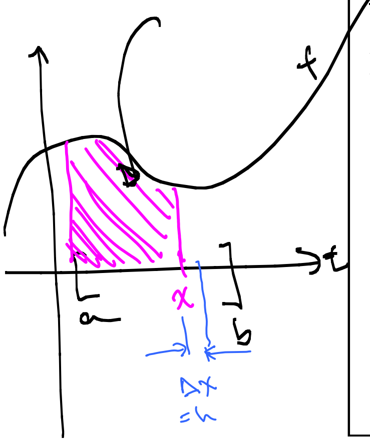
Not continuous at 0

$\boxed{\text{This is an improper integral.}}$

Some improper integrals converge to a finite
finite converge to a number; some do not

For now, if f has an infinite discontinuity anywhere in $[a,b]$, assume that $\int_a^b f(x)\,dx$ does not exist. Some of these integrals do exist….you will learn how to handle such integrals in Calculus 2.

Area $= g(x) = \int_a^x f(t)\,dt$



The Fundamental Theorem of Calculus, Part II:

Let f be continuous on the interval $[a,b]$. Then the function g defined by

$$g(x) = \int_a^x f(t)\,dt, \qquad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

$g'(x) \approx \frac{\Delta g(x)}{\Delta x}$

In other words, $\frac{d}{dx}\left[\underbrace{\int_a^x f(t)\,dt}_{g(x)}\right] = f(x)$.

Example 1: Find the derivative of the function $g(x) = \int_3^x \frac{t^2 - 2t + 4}{t - 2}\,dt$.

here, $f(t) = \frac{t^2-2t+4}{t-2}$

$$g'(x) = \frac{d}{dx}\int_3^x \frac{t^2-2t+4}{t-2}\,dt = \frac{x^2-2x+4}{x-2}$$

Example 2: Find $\frac{d}{dx}\int_{-2}^{\sin x} \sqrt{t^4 + 2}\,dt$.

$A = \text{Area} = \int_{-2}^{\sin x} \sqrt{t^4+2}\,dt$. I want to find $\frac{dA}{dx}$

Let $u = \sin x$

Then $A = \int_{-2}^{u} \sqrt{t^4+2}\,dt$

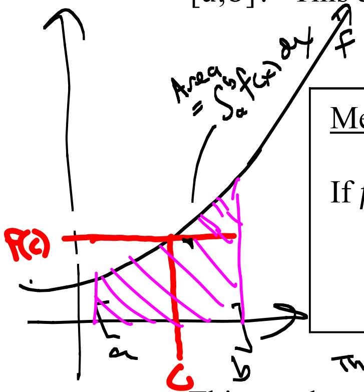
$\frac{dA}{du} = \sqrt{u^4+2}$

Chain rule:

$$\frac{dA}{dx} = \frac{dA}{du} \cdot \frac{du}{dx}$$
$$= \sqrt{u^4+2} \cdot \cos x$$
$$= \boxed{\sqrt{(\sin x)^4+2} \cdot \cos x}$$

The mean (average) value of a function:

On the interval $[a,b]$, a continuous function $f(x)$ will have an average "height" c such that the rectangle with width $b-a$ and height c will have the same area as the area under the curve over $[a,b]$. This c is the *average value of the function f over* $[a,b]$.



> Mean Value Theorem for Integrals:
>
> If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that
>
> $$\int_a^b f(x)\,dx = \overbrace{f(c)}^{\text{height}}\overbrace{(b-a)}^{\text{width}}.$$

The y-value f(c)

~~This number c~~ is called the *average value* of the function *f* on the interval $[a,b]$.

> The *average value* of a continuous function *f* on the interval $[a,b]$ is given by
>
> $$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 8: Find the average value of the function $f(x) = 4x^3 - x^2$ over the interval $[-3,2]$.

$$\text{Area} = \int_{-3}^{2} f(x)\,dx = \int_{-3}^{2}(4x^3 - x^2)\,dx = \left(\frac{4x^4}{4} - \frac{x^3}{3}\right)\Big|_{-3}^{2}$$

$$= \left(x^4 - \frac{x^3}{3}\right)\Big|_{-3}^{2} = \left(2^4 - \frac{2^3}{3}\right) - \left((-3)^4 - \frac{(-3)^3}{3}\right)$$

width $= 2-(-3) = 2+3 = 5$

$$= 16 - \frac{8}{3} - 81 - 9 = -74 - \frac{8}{3} = -\frac{222}{3} - \frac{8}{3} = -\frac{230}{3}$$

(Avg value)(width) = Area

$$\text{Avg value} = \frac{\text{Area}}{2-(-3)} = \frac{\text{Area}}{5} = \left(\frac{1}{5}\right)\left(-\frac{230}{3}\right) = -\frac{230}{15} = \boxed{-\frac{46}{3}}$$

Example 9: Determine the average value of $f(x) = \sin x$ on the interval $[0,\pi]$.

$$\text{Area} = \int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = -\cos(\pi) - (-\cos(0))$$

$$= -(-1) + 1 = 1+1 = 2$$

width $= b - a = \pi - 0 = \pi$

(avg value)(width) = Area

$$\text{so avg value} = \frac{\text{Area}}{\text{width}} = \boxed{\frac{2}{\pi}}$$