5.1: The Natural Logarithmic Function: Differentiation

An algebraic approach to logarithms:

> Definition: $\log_b x = y$ is equivalent to $b^y = x$.
>
> The functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of each other. b is called the *base* of the logarithm.

The logarithm of base *e* is called the *natural logarithm*, which is abbreviated "ln".

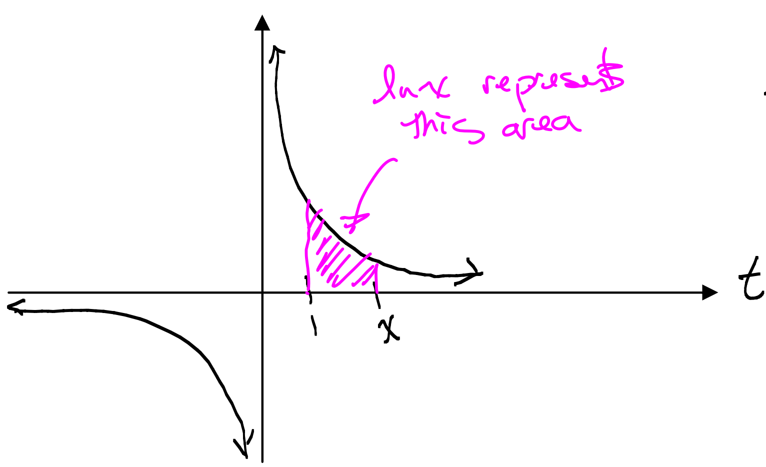
> The natural logarithm:
>
> $\ln x = \log_e x$.
>
> Therefore $\ln x = y$ is equivalent to $e^y = x$ and the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverses of each other.

A calculus approach to the natural logarithm:

> The natural logarithm function is defined as
>
> $$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

$\ln x = \int_1^x \frac{1}{t}\, dt$

$x > 0$



$f(t) = \frac{1}{t}$

$y = \frac{1}{t}$

For $x > 1$, $\ln x$ can be interpreted as the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to $t = x$.

so $\ln(x)$ is undefined for $x<0$

Note: The integral is not defined for $x<0$. (improper integral)

For $x=1$, $\ln x = \int_1^1 \frac{1}{t}\,dt = 0$. So $\ln(1)=0$

For $x<1$, $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$.

Recall:

> The Fundamental Theorem of Calculus, Part II:
>
> Let *f* be continuous on the interval $[a,b]$. Then the function g defined by
>
> $$g(x) = \int_a^x f(t)\,dt\,, \qquad a \le x \le b$$
>
> is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Apply the Fundamental Theorem of Calculus to the function $f(t) = \frac{1}{t}$. $g(x) = \ln x = \int_1^x \frac{1}{t}\,dt$

$$\frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}$$

This means that $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

> The Derivative of the Natural Logarithmic Function
>
> $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Laws of Logarithms:

If *x* and *y* are positive numbers and *r* is a rational number, then:

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

   Note: This also gives us $\ln\left(\frac{1}{x}\right) = -\ln x$.

3. $\ln\left(x^r\right) = r\ln x$

Example 1: Expand $\ln\left(\dfrac{x^3\sqrt{x+5}}{x^2+4}\right)$.

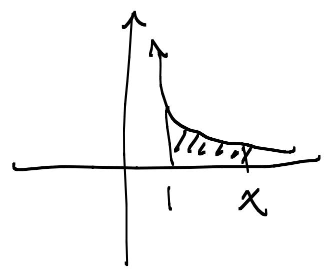
$$\ln\left(\frac{x^3(x+5)^{1/2}}{x^2+4}\right) = \ln\left[x^3(x+5)^{1/2}\right] - \ln(x^2+4)$$

$$= \ln x^3 + \ln(x+5)^{1/2} - \ln(x^2+4)$$

$$= \boxed{3\ln x + \tfrac{1}{2}\ln(x+5) - \ln(x^2+4)}$$

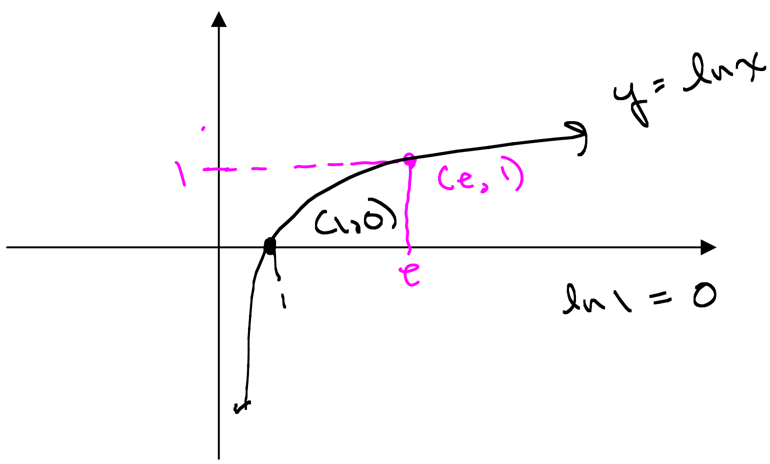
The graph of $y = \ln x$:

It can be shown that $\lim\limits_{x\to\infty} \ln x = \infty$ and that $\lim\limits_{x\to 0^+} (\ln x) = -\infty$.



For $x > 0$, $\dfrac{dy}{dx} = \dfrac{1}{x} > 0$ so $y = \ln x$ is increasing on $(0,\infty)$.

For $x > 0$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2} < 0$ so $y = \ln x$ is concave down on $(0,\infty)$.



Because $\ln 1 = 0$ and $y = \ln x$ is increasing to arbitrarily large values $\left(\lim\limits_{x\to\infty} \ln x = \infty\right)$, the Intermediate Value Theorem guarantees that there is a number *x* such that $\ln x = 1$. That number is called *e*.

$$e \approx 2.71828182845904523536$$

(*e* is in irrational number—it cannot be written as a decimal that ends or repeats.)

Example 2: Find $\frac{dy}{dx}$ for $y = \ln(2x^5 + 3x)$.

$$\frac{dy}{dx} = \frac{1}{2x^5+3x} \cdot \frac{d}{dx}(2x^5+3x) = \frac{1}{2x^5+3x} \cdot (10x^4+3)$$

$$= \frac{10x^4+3}{2x^5+3x} = \boxed{\frac{10x^4+3}{x(2x^4+3)}}$$

Note: $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ or, written another way, $\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$.

Example 3: Determine $\frac{d}{dx}(\ln(\cos x))$.

Example 4: Find the derivative of $f(x) = \frac{1}{\ln x}$.

Example 5: Find the derivative of $f(x) = x^2 \ln x$.

Example 6: Find the derivative of $y = \frac{\ln x}{4x}$.

Example 7: Find the derivative of $g(t) = \ln(7t)$.

Example 8: Determine the derivative of $f(x) = \dfrac{\ln 6x}{(x+4)^5}$.

$$f'(x) = \frac{(x+4)^5 \frac{d}{dx}(\ln(6x)) - (\ln(6x)) \frac{d}{dx}(x+4)^5}{(x+4)^{10}}$$

$$= \frac{(x+4)^5 \left(\frac{1}{6x}\right)(6) - (\ln(6x))(5)(x+4)^4(1)}{(x+4)^{10}}$$

$$= \frac{(x+4)^5 \left(\frac{1}{x}\right) - 5(x+4)^4 \ln(6x)}{(x+4)^{10}}$$

$$\frac{(x+4)^4\left[(x+4)\left(\frac{1}{x}\right) - 5\ln(6x)\right]}{(x+4)^{10}} = \frac{(x+4)\left(\frac{1}{x}\right) - 5\ln(6x)}{(x+4)^6}\left(\frac{x}{x}\right)$$

$$= \boxed{\frac{x+4-5x\ln(6x)}{x(x+4)^6}}$$

Logarithmic differentiation:

To differentiate $y = f(x)$:

1. Take the natural logarithm of both sides.

2. Use the laws of logarithms to expand.

3. Differentiate implicitly with respect to x.

4. Solve for $\dfrac{dy}{dx}$.

Example 9: Use logarithmic differentiation to find the derivative of

$$y=(x^2+2)^5(2x+1)^3(6x-1)^2.$$

$$\ln y = \ln\left[(x^2+2)^5(2x+1)^3(6x-1)^2\right]$$

$$\ln y = \ln(x^2+2)^5 + \ln(2x+1)^3 + \ln(6x-1)^2$$

$$\ln y = 5\ln(x^2+2) + 3\ln(2x+1) + 2\ln(6x-1)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[5\ln(x^2+2) + 3\ln(2x+1) + 2\ln(6x-1)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 5\left(\frac{1}{x^2+2}\right)(2x) + 3\left(\frac{1}{2x+1}\right)(2) + 2\left(\frac{1}{6x-1}\right)(6)$$

$$\frac{dy}{dx} = y\left[\frac{10x}{x^2+2} + \frac{6}{2x+1} + \frac{12}{6x-1}\right] = (x^2+2)^5(2x+1)^3(6x-1)^2\left[\frac{10x}{x^2+2} + \frac{6}{2x+1} + \frac{12}{6x-1}\right]$$

Example 10: Find y' for $y=\dfrac{(x^3+1)^4\sin^2 x}{\sqrt[3]{x}}$.

$$\ln y = \ln\left[\frac{(x^3+1)^4(\sin x)^2}{x^{1/3}}\right]$$

$$\ln y = 4\ln(x^3+1) + 2\ln(\sin x) - \frac{1}{3}\ln x$$

Ex 11: $f(x) = x^{\tan x}$. Find $f'(x)$.

$$y = x^{\tan x}$$

$$\ln y = \ln\left(x^{\tan x}\right)$$

$$\ln y = (\tan x)(\ln x)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left((\tan x)(\ln x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = (\tan x)\left(\frac{1}{x}\right) + (\ln x)(\sec^2 x)$$