5.2: The Natural Logarithmic Function: Integration

Using the derivative of the natural logarithmic function to obtain an antiderivative:

Example 1: Find the derivative of $g(x) = \ln|x|$.

Note: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$g(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$ Note: $\ln|x|$ is undefined for $x = 0$

$g'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) & \text{if } x < 0 \end{cases} \Rightarrow g'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases}$

$\boxed{\text{So } g'(x) = \frac{1}{x} \text{ for all } x \text{ except } 0.}$

Note that $f(x) = \ln x$ has the same derivative as $g(x) = \ln|x|$.

Therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$. This means that $f(x) = \ln|x|$ is an antiderivative of $F(x) = \frac{1}{x}$.

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + c}$$

Recall: The power rule for integrals $\int x^n dx = \frac{x^{n+1}}{n+1}$ had a restriction: ~~$n \neq -1$~~ $n \neq -1$. Now we can handle this case.

Example 2: Determine $\int \frac{x^2}{x^3+4}\,dx$.

$$\int \frac{x^2}{x^3+4}\,dx = \int \underbrace{x^2 \, dx}_{\frac{1}{3}du} \underbrace{\left(\frac{1}{x^3+4}\right)}_{\frac{1}{u}} = \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|u| + c$$

$$= \boxed{\frac{1}{3}\ln\left|x^3+4\right| + C}$$

$u = x^3+4$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

check: $\frac{d}{dx}\left(\frac{1}{3}\ln(x^3+4)\right)$

$= \frac{1}{\cancel{3}} \cdot \frac{1}{x^3+4} \cdot \cancel{3}x^2$

$= \frac{x^2}{x^3+4}$ ✓

Example 3: Determine $\int \frac{7}{2-5x}\,dx$.

$$\int \frac{7}{2-5x}\,dx = 7\int \underbrace{\frac{1}{2-5x}}_{\frac{1}{u}} \underbrace{dx}_{-\frac{1}{5}du}$$

$$= 7\left(-\frac{1}{5}\right)\int \frac{1}{u}\,du$$

$$= -\frac{7}{5}\ln|u| + c = \boxed{-\frac{7}{5}\ln|2-5x| + C}$$

$u = 2-5x$

$\frac{du}{dx} = -5$

$du = -5\,dx$

$-\frac{1}{5}du = dx$

Example 4: Determine $\int_2^5 \frac{1}{3x}\,dx$.

$$\int_2^5 \frac{1}{3x}\,dx = \frac{1}{3}\int_2^5 \frac{1}{x}\,dx = \frac{1}{3}\ln|x| \Big[_2^5$$

$$= \frac{1}{3}\ln|5| - \frac{1}{3}\ln|2|$$

$$= \boxed{\frac{1}{3}(\ln 5 - \ln 2)} = \boxed{\frac{1}{3}\ln\left(\frac{5}{2}\right)}$$

Example 5: Determine $\int \frac{x^7 - x + 3x^4}{x^5}\,dx$.

$$\int\left(\frac{x^7}{x^5} - \frac{x}{x^5} + \frac{3x^4}{x^5}\right)dx = \int\left(x^2 - \frac{1}{x^4} + \frac{3}{x}\right)dx$$

$$= \int\left(x^2 - x^{-4} + 3\left(\frac{1}{x}\right)\right)dx = \int x^2\,dx - \int x^{-4}\,dx + 3\int \frac{1}{x}\,dx$$

$$= \frac{x^3}{3} - \frac{x^{-3}}{-3} + 3\ln|x| + C$$

$$= \boxed{\frac{x^3}{3} + \frac{1}{3x^3} + 3\ln|x| + C}$$

Example 6: Find $\int \frac{(\ln x)^4}{x}\,dx$.

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\,dx$

$$\int \underbrace{\frac{1}{x}\cdot \underbrace{(\ln x)^4}_{u^4}\,dx}_{du} = \int u^4\,du$$

$$= \frac{u^5}{5} + C$$

$$= \boxed{\frac{(\ln x)^5}{5} + C}$$

Example 7: Find $\int \frac{\ln(3x)}{x}\,dx$.

$u = \ln(3x)$

$\frac{du}{dx} = \frac{1}{3x}(3)$

$= \frac{1}{x}$

$du = \frac{1}{x}\,dx$

Try $u = 3x$

$\frac{du}{dx} = 3$

$du = 3\,dx$

No.

$$\int \frac{\ln(3x)}{x}\,dx = \int \underbrace{(\ln(3x))}_{u} \underbrace{\left(\frac{1}{x}\right)dx}_{du}$$

$$= \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{(\ln(3x))^2}{2} + C}$$

Example 8: Find $\int \frac{x}{x^2-8}\,dx$.

$u = x^2 - 8$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\int \frac{x}{x^2-8}\,dx = \int \underbrace{x \left(\frac{1}{\underbrace{x^2-8}_{\frac{1}{u}}}\right) dx}_{\frac{1}{2}du} = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \boxed{\frac{1}{2}\ln|x^2-8| + C}$$

Check: $\frac{d}{dx}\left(\frac{1}{2}\ln(x^2-8)\right)$

$= \frac{1}{2}\cdot\frac{1}{x^2-8}\cdot 2x = \frac{x}{x^2-8}$

✓ ok

Example 9: Find $\int \frac{4x^2-5x-12}{x^2-3}\,dx$.

Long Division:

$$\begin{array}{r} 4 \\ x^2-3 \overline{)\,4x^2-5x-12} \\ -(4x^2 \qquad -12) \\ \hline -5x \quad +0 \end{array}$$

$$\text{Integral} = \int \left(4 - \frac{5x}{x^2-3}\right) dx$$

$$= \int 4\,dx - 5\int \underbrace{\frac{x}{\underbrace{x^2-3}_{1/u}}\,dx}_{\frac{1}{2}du}$$

$$= 4x - 5\left(\tfrac{1}{2}\right)\int \frac{1}{u}\,du$$

$$= 4x - \frac{5}{2}\ln|u| + C = \boxed{4x - \frac{5}{2}\ln|x^2-3| + C}$$

$u = x^2 - 3$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

Example 10: Find $\int \frac{4x^2-7x+1}{2x-3}\,dx$.

$$\frac{4x^2}{2x} \quad \frac{-x}{2x}$$

$$\begin{array}{r} 2x - \frac{1}{2} \\ 2x-3 \overline{\big) 4x^2-7x+1} \\ -(4x^2-6x) \\ \hline -x+1 \\ -(-x+3/2) \\ \hline -1/2 \end{array}$$

Integral $= \int \left(2x - \frac{1}{2} - \frac{1/2}{2x-3}\right) dx$

$= \int 2x\,dx - \int \frac{1}{2}\,dx - \frac{1}{2}\int \underbrace{\frac{1}{2x-3}}_{1/u}\, \underbrace{dx}_{\frac{1}{2}du}$

$= \frac{2x^2}{2} - \frac{1}{2}x - \frac{1}{2}\int \frac{1}{u}\left(\frac{1}{2}\right)du$

$= x^2 - \frac{1}{2}x - \frac{1}{4}\int \frac{1}{u}\,du$

$= x^2 - \frac{1}{2}x - \frac{1}{4}\ln|u| + c = \boxed{x^2 - \frac{1}{2}x - \frac{1}{4}\ln|2x-3| + c}$

$u = 2x-3$

$\frac{du}{dx} = 2$

$du = 2\,dx$

$\frac{1}{2}du = dx$

Integrating the remaining trigonometric functions:

Example 11: Determine $\int \tan x\,dx$.

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \sin x \cdot \underbrace{\frac{1}{\cos x}}_{\frac{1}{u}} \cdot dx$, with $\sin x\,dx = -du$

$= -\int \frac{1}{u}\,du$

$= -\ln|u| + c = \boxed{-\ln|\cos x| + c}$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$-du = \sin x\,dx$

Example 12: Determine $\int \cot x\,dx$.

$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \cos x \cdot \underbrace{\frac{1}{\sin x}}_{\frac{1}{u}}\,dx$, with $\cos x\,dx = du$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$= \int \frac{1}{u}\,du = \ln|u| + c$

$= \boxed{\ln|\sin x| + c}$

Check: $\frac{d}{dx}\left(\ln(\sin x)\right)$

$= \frac{1}{\sin x}\frac{d}{dx}(\sin x)$

$= \frac{1}{\sin x}\cdot\cos x = \cot x$ ✓ok

Example 13: Determine $\int \sec x\, dx$.

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$$= \int \underbrace{(\sec^2 x + \sec x \tan x)\, dx}_{du} \underbrace{\left(\frac{1}{\sec x + \tan x}\right)}_{\frac{1}{u}}$$

$u = \sec x + \tan x$

$\frac{du}{dx} = \sec x \tan x + \sec^2 x$

$du = (\sec x \tan x + \sec^2 x)dx$

$$= \int \frac{1}{u}\, du = \ln|u| + c = \boxed{\ln|\sec x + \tan x| + c}$$

Example 14: Determine $\int \csc x\, dx$.

$$\int \csc x\, dx = \int \csc x \cdot \frac{\csc x + \cot x}{\csc x + \cot x}\, dx$$

$u = \csc x + \cot x$

$\frac{du}{dx} = -\csc x \cot x - \csc^2 x$

$-du = \csc x \cot x + \csc^2 x\, dx$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx$$

$$= \int \underbrace{\frac{1}{\csc x + \cot x}}_{\frac{1}{u}} \cdot \underbrace{\frac{\csc^2 x + \csc x \cot x}{1}\, dx}_{-du}$$

$$= -\int \frac{1}{u}\, du$$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\csc x + \cot x| + C}$$