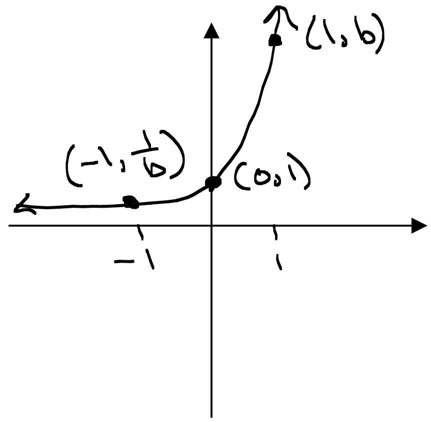
5.4: Exponential Functions: Differentiation and Integration

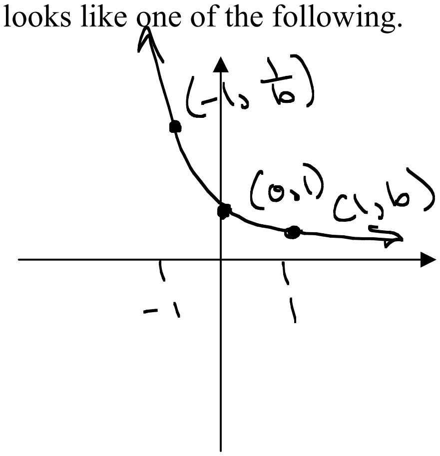
Short Review:

An *exponential* function takes the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

For any exponential function $f(x) = b^x$, the graph looks like one of the following.



$b > 1$



$0 < b < 1$

Notice:

- Domain is $(-\infty, \infty)$.
- Range is $(0, \infty)$.
- Horizontal asymptote is $y = 0$.
- Always passes through the points $(-1, \frac{1}{b})$ $(1, b)$.

The natural exponential function:

The number *e* can be defined in several ways.

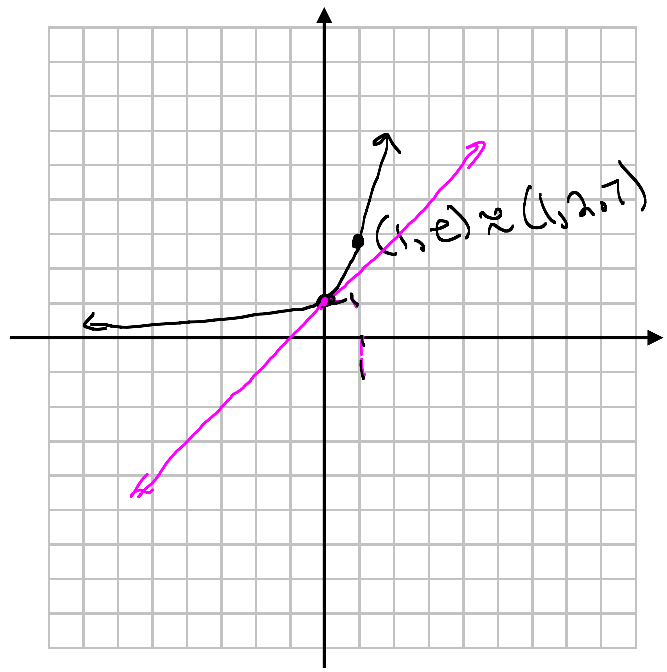
One definition of the number *e*:

e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$$\boxed{e \approx 2.718281828459}$$

The slope of the tangent line at the point $(0,1)$ is equal to 1.

The graph of $f(x) = e^x$**:**



Another definition of the number e:

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \quad \text{or, equivalently,} \quad e = \lim_{x \to 0} (1 + x)^{1/x}$$

$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

Let $h = \frac{1}{x} \Rightarrow$

$e = \lim_{h \to 0} (1 + h)^{1/h}$

Derivatives of exponential functions:

$$\boxed{\frac{d}{dx}\left(e^x\right) = e^x}$$

<u>Example 1:</u> Find the derivative of $f(x) = -7e^x$.

$f'(x) = \boxed{-7e^x}$

Example 2: Find the derivative of $f(x) = 5\sqrt{e^x + 7}$.

$f(x) = 5(e^x+7)^{1/2}$

$f'(x) = 5\left(\frac{1}{2}\right)(e^x+7)^{-1/2}\frac{d}{dx}(e^x+7) = \frac{5}{2}(e^x+7)^{-1/2}(e^x)$

Example 3: Find the derivative of $f(x) = e^x \sin x$.

$f'(x) = e^x \cos x + (\sin x)e^x$ [product rule]

$= e^x \cos x + e^x \sin x = \boxed{e^x(\cos x + \sin x)}$

Example 4: Find the derivative of $g(x) = e^{-7x} + 2x^3 - 4e$.

$g'(x) = e^{-7x}\frac{d}{dx}(-7x) + 6x^2 + 0$

$= e^{-7x}(-7) + 6x^2 = \boxed{-7e^{-7x} + 6x^2}$

Example 5: Find the derivative of $y = e^{x^2+4x}$.

$y = e^{x^2+4x}$

$\frac{dy}{dx} = e^{x^2+4x}\frac{d}{dx}(x^2+4x)$

$= \boxed{e^{x^2+4x}(2x+4)}$

Example 6: Find the derivative of $f(x) = \cos(e^x - x)$.

$f'(x) = -\sin(e^x - x)\frac{d}{dx}(e^x - x)$

$= \left[-\sin(e^x - x)\right]\left[e^x - 1\right] = \boxed{-(e^x-1)\sin(e^x - x)}$

Example 7: Find the equation of the tangent line to the graph of $f(x) = (e^x + 2)^2$ at the point $(0,9)$.

$f'(x) = 2(e^x+2)\frac{d}{dx}(e^x+2)$

$= 2(e^x+2)(e^x+0) = 2e^x(e^x+2)$

slope: $m = f'(0) = 2e^0(e^0+2) = 2(1)(1+2) = 2(3) = 6$

Check/find point: $f(0) = (e^0+2)^2 = (1+2)^2 = 3^2 = 9$ ✓

$y - y_1 = m(x - x_1)$ or $y = mx + b$

$y - 9 = 6(x - 0)$ $\boxed{y = 6x + 9}$

$y - 9 = 6x$

$\boxed{y = 6x + 9}$

Integration of exponential functions:

$$\int e^x dx = e^x + c$$

Example 8: Determine $\int (x^2 - 5e^x)\, dx$

$$\int (x^2 - 5e^x)\, dx = \boxed{\frac{x^3}{3} - 5e^x + c}$$

Example 9: Find $\int e^{5t} dt$.

$$\int \underbrace{e^{5t}}_{e^u} \underbrace{dt}_{\frac{1}{5}du}$$

$$= \frac{1}{5}\int e^u du = \frac{1}{5}e^u + c$$

$$= \boxed{\frac{1}{5}e^{5t} + c}$$

$u = 5t$

$\frac{du}{dt} = 5$

$du = 5\, dt$

$\frac{1}{5} du = dt$

Example 10: Find $\int_1^3 e^{2x-3} dx$.

$$\int_1^3 \underbrace{e^{2x-3}}_{e^u} \underbrace{dx}_{\frac{1}{2}du}$$

$u = 2x - 3$

$du = 2\, dx$

$\frac{1}{2} du = dx$

$x = 1 \Rightarrow u = 2(1) - 3 = -1$

$x = 3 \Rightarrow u = 2(3) - 3 = 3$

$$= \frac{1}{2}\int_{-1}^{3} e^u du$$

$$= \frac{1}{2} e^u \Big|_{u=-1}^{u=3} = \boxed{\frac{1}{2}\left[e^3 - e^{-1}\right]} = \boxed{\frac{e^3}{2} - \frac{1}{2e}}$$

Example 11: Find $\int te^{t^2}\,dt$.

$u = t^2$

$\frac{du}{dt} = 2t$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$$\int \underbrace{t}\,\underbrace{e^{t^2}}_{e^u}\,\underbrace{dt}_{\frac{1}{2}du} = \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + C$$

$$= \boxed{\frac{1}{2}e^{t^2} + C}$$

Example 12: Determine $\int \frac{e^x}{\sqrt[3]{e^x+1}}\,dx$.

$u = e^x + 1$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

$$\int e^x\,\underbrace{(e^x+1)^{-1/3}}_{u^{-1/3}}\,dx = \int u^{-1/3}\,du$$

$$= \frac{u^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C = \frac{u^{2/3}}{2/3} + C = \frac{3u^{2/3}}{2} + C$$

$$= \boxed{\frac{3}{2}(e^x+1)^{2/3} + C}$$

Example 13: Determine $\int \frac{e^x - e^{-x}}{e^{3x}}\,dx$

$$\int\left(\frac{e^x}{e^{3x}} - \frac{e^{-x}}{e^{3x}}\right)dx = \int\left(e^{x-3x} - e^{-x-3x}\right)dx$$

$$= \int\left(e^{-2x} - e^{-4x}\right)dx = \int e^{-2x}\,dx - \int e^{-4x}\,dx$$

1st one:

$u = -2x$

$du = -2\,dx$

$-\frac{1}{2}du = dx$

2nd one:

$u_2 = -4x$

$du_2 = -4\,dx$

$-\frac{1}{4}du_2 = dx$

$$= -\frac{1}{2}\int e^u\,du - \left(-\frac{1}{4}\right)\int e^{u_2}\,du_2$$

$$= -\frac{1}{2}e^u + \frac{1}{4}e^{u_2} + C$$

$$= \boxed{-\frac{1}{2}e^{-2x} + \frac{1}{4}e^{-4x} + C}$$