12.1: Confidence Intervals for One Population Proportion

Recall: A *parameter* is a numerical summary of a population; a *statistic* is a numerical summary of a sample. (For example, the population mean and population standard deviation are parameters; the sample mean and sample standard deviation are statistics.)

The *sampling distribution of a statistic* is the probability distribution of all possible values for that statistic computed from all possible samples of fixed size *n*.

The sampling distribution of the sample proportion:

In this section, the parameter we are interested in is the *population proportion*, usually denoted *p*.

The proportion is the percentage *p* (in decimal form) of the population that possesses some characteristic of interest.

For example, we may be interested in the proportion of children who have a certain medical condition, the proportion of U.S. citizens who received a tax refund, the proportion of students at a certain high school that decide to go to college, or the proportion of nurse candidates who pass the nursing licensure exam.

The *sampling distribution of the sample proportion* is the probability distribution of all possible values for the sample proportion, denoted $\hat{p}$, computed from all possible samples of fixed size *n*.

p-hat $\hat{p}$

If *x* is the number of data points in a sample of size *n* that have the characteristic of interest, then the sample proportion is

$$\hat{p} = \frac{x}{n}.$$

In the same manner as for the sample mean, we use the sample proportion $\hat{p}$ to make inferences about the population proportion *p*.

Shape, mean and standard deviation of the sampling distribution of the sample proportion:

Sampling distribution of the sample proportion:

Suppose random samples of size n are taken from a population with population proportion p.

Also suppose that the sample size is small compared to the size of the population.
(Rule of thumb: The sample must be less than 5% of the population size; otherwise we must use a finite population correction factor, which is beyond the scope of this class.)

Then:

The shape of the sampling distribution of $\hat{p}$ is approximately normal, provided that the sample sizes are sufficiently large.

Rule of thumb: to assume the sample proportion is normally distributed, we need both $np \geq 5$ and $n(1-p) \geq 5$.

(each cell in the table must have at least 5 entries)

The mean of the sampling distribution of $\hat{p}$ is $\mu_{\hat{p}} = p$.

The standard deviation of the sampling distribution of : $\hat{p}$ is $\sigma_{\hat{p}} = \sqrt{\dfrac{p(1-p)}{n}}$.

(standard error)

letting $q = 1-p$ this becomes

$\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}}$

Point estimates for the population proportion:

Recall:

Definition: A *point estimate* is the value of a statistic that estimates the value of a parameter.

Definition: A confidence interval for an unknown parameter is an interval of numbers generated by a point estimate for that parameter.

Definition: The *confidence level* (usually given as a percentage) represents how confident we are that the confidence interval contains the parameter.

If a large number of samples is obtained, and a separate point estimate and confidence interval are generated from each sample, then a 95% confidence level indicates that 95% of all these confidence intervals contain the population parameter.

A confidence interval is obtained by placing a *margin of error* on either side of the point estimate of the parameter.

In other words, the confidence interval consists of: Point estimate $\pm$ margin of error

The point estimate of the population proportion p is the sample proportion $\hat{p}$.
The point estimate of the mean of the sampling distribution of the sample proportions is $\mu_{\hat{p}} = \hat{p}$.

The point estimate of the standard deviation of the sampling distribution of the sample proportions is

$$\sigma_{\hat{p}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$ (estimate for the standard error)

So, for every sample, the sample proportion will be in the center of the confidence interval. If we use E to indicate the margin of error, the confidence interval is $\hat{p} \pm E$, or $\left(\hat{p}-E, \hat{p}+E\right)$

If we use the sample proportion $\hat{p}$ as a starting point, we should be able to write the confidence interval as $\left(\hat{p}-z_c\sigma_{\hat{p}}, \hat{p}+z_c\sigma_{\hat{p}}\right)$, where $\sigma_{\hat{p}}$ is the standard deviation of the sampling distribution of the sample proportions, and z_c is a multiplier that tells us how many standard deviations (of the sampling distribution of the sample proportions) lie between the sample proportion $\hat{p}$ and the edge of the confidence interval. We call this z_c the *critical value* for a *z*-score in the sampling distribution of the sample proportions.

Constructing the confidence interval for the proportion:

Procedure:

1. Verify that $np \geq 5$ and $n(1-p) \geq 5$ and that the sample is no more than 5% of the population.

2. Determine the confidence level, $1-\alpha$.

3. Determine the critical value $z_{\alpha/2}$ (using the standard normal table).

4. Use the sample proportion to estimate the standard deviation of the sampling distribution of the sample proportions:

$$\sigma_{\hat{p}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

5. Multiply the critical value $z_{\alpha/2}$ by the estimated standard deviation $\sigma_{\hat{p}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ to obtain the margin of error.

6. Add and subtract the margin of error from the sample proportion to obtain the lower and upper bounds of the confidence interval:

   Lower bound: $\hat{p} - z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

   Upper bound: $\hat{p} + z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Useful critical values of z:
(You can use these instead of looking in the table every time).

Confidence Level	α	Area in each tail, $\alpha/2$	Critical value $z_{\alpha/2}$
90%	0.10	0.05	1.645
95%	0.05	0.025	1.96
99%	0.01	0.005	2.575

Check assumptions: Eat lots of PB: 173 Don't eat lots of PB: $537-173=364$ Both ≥ 5 so $\hat{p}$ is normally distributed

Example 1: In a random sample of 537 Americans, 173 indicated that they frequently ate peanut butter. Construct and interpret the 90% and the 95% confidence intervals for the proportion of Americans who frequently eat peanut butter.

Sample info

$n = 537$

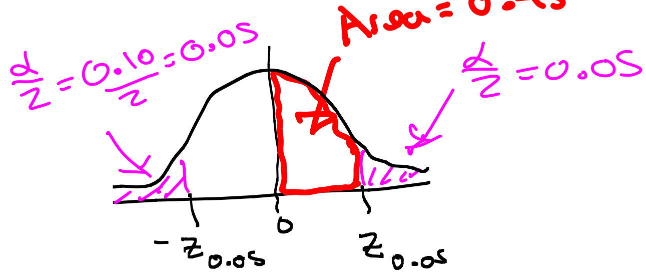
$\hat{p} = \frac{x}{n} = \frac{173}{537} \approx 0.32216 \approx 0.322$

$\hat{q} = 1-\hat{p} = 1-0.322 = 0.678$

std. error:

$$\sigma_{\hat{p}} \approx \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{\hat{p}\hat{q}}{n}} = \sqrt{\frac{0.322(0.678)}{537}}$$

≈ 0.020163 (store in calculator)



Construct 90% confidence interval:

$\alpha = 1-0.90 = 0.10$

Look up Area = 0.45 on normal table. Closest areas are 0.4495 and 0.4505, corresponding to $z=1.64$ and $z=1.65$. So critical value is $z_{0.05} = 1.645$

Lower bound: $\hat{p} - z_{0.05}\sigma_{\hat{p}}$

$= 0.322 - 1.645(0.02163) = 0.289$

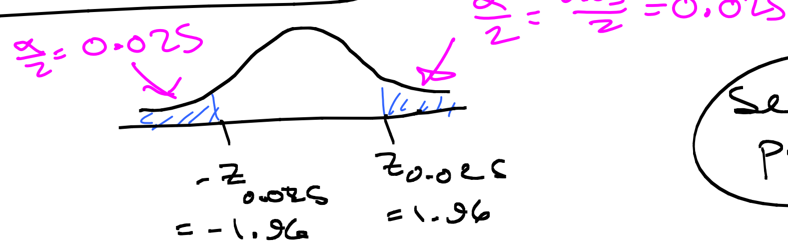
Upper bound: $\hat{p} + z_{0.05}\sigma_{\hat{p}}$

$= 0.322 + 1.645(0.02163) = 0.355$

$\boxed{\text{90\% CI: } (0.289, 0.355)}$

Construct 95% CI:

$\alpha = 1-0.95 = 0.05$



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Lower bound: $\hat{p} - z_{0.025}\sigma_{\hat{p}} = 0.322 - 1.96(0.02163) \approx 0.283$

Upper bound: $\hat{p} + z_{0.025}\sigma_{\hat{p}} = 0.322 + 1.96(0.02163) \approx 0.362$

$\boxed{95\% \text{ CI: } (0.283, 0.362)}$

Sample size needed to estimate the population proportion within a given margin of error:

When constructing a confidence interval about the sample proportion $\hat{p}$, the margin of error is

$$z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

skip estimate the sample size

We can solve this for n:

In order to calculate the n needed, we need an educated guess for the population proportion p. If such an educated guess is available (perhaps from a prior study), we can use the above formula to calculate n. If not, we use the very conservative assumption that $\hat{p} = 0.5$, which gives us the maximum possible value for $\hat{p}(1-\hat{p})$, which is $(0.5)(0.5) = 0.25$.

<u>Required sample size for estimation of the population proportion</u>:

a) For a specified α associated with a confidence level, the sample size required to estimate the population proportion within a margin of error E is

$$n = \hat{p}_g(1-\hat{p}_g)\left(\frac{z_{\alpha/2}}{E}\right)^2,$$

where $\hat{p}_g$ is an educated guess for the population proportion p.

b) If you know a likely range of values for the sample proportion, choose the value in that range that is closest to 0.5. Use this value as the educated guess $\hat{p}_g$ in the above formula. (The above formula will be at its maximum when $\hat{p}_g = 0.5$. Thus a larger sample is required when $\hat{p}_g$ is close to 0.5, compared to when $\hat{p}_g$ is further away. To be sure we have a big enough sample, we look at all the possible values for $\hat{p}$ and choose the one closest to 0.5.)

c) If no estimate for the population proportion is available, we should use a sample size of at least

$$n = 0.25\left(\frac{z_{\alpha/2}}{E}\right)^2.$$

In all cases, because the calculated n is considered a minimum threshold, we round the calculated value of n <u>up</u> to the nearest whole number *above*.

<u>Example 2:</u> A pollster wishes to estimate the percentage of likely voters who support Candidate A. Based on earlier polls, the pollster expects the candidate's level of support to be approximately 38%. What sample size should be obtained if the pollster wishes to estimate the candidate's support level within a margin of error of 3 percentage points, with 95% confidence?

Example 3: a) Estimate the minimum sample size required to estimate the population proportion within a margin of error of 0.02, if the proportion is expected to be between 0.1 and 0.3. Use a confidence level of 95%.
b) Suppose the sample size from part (a) is obtained, and that the proportion of the characteristic of interest turns out to be 0.25. Construct the 95% confidence interval for the population proportion. What is the margin of error?

Example 4: Estimate the minimum sample size required to estimate the population proportion within a margin of error of 0.03, if you have no idea what the proportion will turn out to be.