4.2: Systems of Linear Equations and Augmented Matrices
4.3: Gauss-Jordan Elimination

We have discussed using the substitution and elimination methods of solving a system of linear equations in two variables. While these techniques also work for larger systems, they can be lengthy and cumbersome. Now, we will look at matrix techniques, which can easily be computerized, to solve any size system.

Matrices:

A *matrix* is a rectangular array of numbers written within brackets.

Examples: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 6 & 9 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 9 & 12 \\ -2 & 4 & 8 & -5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$, $E = \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}$, $F = \begin{bmatrix} 7 \end{bmatrix}$

Each number is an *element*. The *size* is expressed as $m \times n$, where m is the number of rows and n is the number of columns.

Special matrices:
- *Row Matrix:* one row (matrices D and F in the example above).
- *Column Matrix:* one column (matrices E and F in the example above).
- *Square Matrix:* same number of rows as columns (matrices C and F in the example above).

The *position* of an element is given by the row and column in the matrix where the element is.

Example 1: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ -1 & 0 & -8 \\ 7 & -3 & 9 \end{bmatrix}$. Element a_{ij} is in row i and column j.

The *principal diagonal* is the collection of elements a_{ii}. (elements with row number and column number the same).

Example 2: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ -1 & 0 & -8 \\ 7 & -3 & 9 \end{bmatrix}$.

Augmented matrices:

In the solution of systems of linear equations, we will use *augmented matrices*.

Example 3: $\begin{cases} 2x_1 + x_2 = 7 \\ 3x_1 + 2x_2 = 12 \end{cases}$

Each equation gives us a row in the augmented matrix. The coefficients and the constants are the elements.

Augmented matrix: $\begin{bmatrix} 2 & 1 & 7 \\ 3 & 2 & 12 \end{bmatrix}$

Each column to the left of the vertical bar represents the coefficients of a particular variable. The column to the right represents the "constants" in the equations. These constants should be written on the right-hand side of the equals sign.

Example 4: Write the augmented matrix for the system.

$2x - 3y + z = 9$

$-x + 8z = -2$

$4x - 7y + 6z = 0$

Example 5: Give the system represented by $\begin{bmatrix} -8 & 0 & 7 & 1 \\ 5 & 3 & -6 & 2 \\ 0 & 2 & -1 & -3 \end{bmatrix}$.

Example 6: Solve the system represented by this matrix. $\begin{bmatrix} 1 & 4 & 7 & | & 1 \\ 0 & 1 & -6 & | & 2 \\ 0 & 0 & 1 & | & -3 \end{bmatrix}$.

This is called *back-substitution*.

Row echelon form:

A matrix with 1's down the principal diagonal and 0's below the diagonal is said to be in *row-echelon form*. We can put any matrix in row-echelon form using row operations to produce equivalent matrices.

Row equivalency:

Two augmented matrices are *row-equivalent* if they correspond to equivalent linear systems (i.e. they have the same solution set). This is denoted by the symbol ~ (e.g. $A \sim B$).

To solve a system using augmented matrices, we use the following *row operations* to produce new row-equivalent matrices.

Operations that produce row-equivalent matrices:

An augmented matrix is transformed into a row-equivalent matrix by performing any of the following:

    a. Two rows are interchanged $\left(R_i \leftrightarrow R_j \right)$.

    b. A row is multiplied by a nonzero constant $\left(kR_i \rightarrow R_i \right)$.

    c. A constant multiple of one row is added to another row $\left(kR_j + R_i \rightarrow R_i \right)$.

Note: The arrow $\rightarrow$ means "replaces".

Gaussian Elimination with Back-Substitution:

We use the row operations to write the augmented matrix in row-echelon form. Then use back-substitution to solve the system. Don't forget to check your answers!

Example 7:   Solve the following system using augmented matrices.

$$\begin{cases} -3x_1 + 2x_2 = 11 \\ 8x_1 - 5x_2 = -21 \end{cases}$$

Recall: Not every system has a unique solution. Some have infinitely many solutions and some have no solution at all.

Here's how to tell when these situations occur: (a and b are real numbers).

- No solution (inconsistent): you get $\begin{bmatrix} 1 & a & | & b \\ 0 & 0 & | & c \end{bmatrix}$, where $c \neq 0$.

- Infinitely many solutions (consistent and dependent): you get $\begin{bmatrix} 1 & a & | & b \\ 0 & 0 & | & 0 \end{bmatrix}$. In other words, you get a zero row, because both equations represent the same line.

Reduced matrices:

Row-Echelon Form:
  All rows consisting entirely of zeros (if any) are at the bottom.
  The first nonzero entry in each row is 1 (called a leading 1).
  Each leading 1 appears to the right of the leading 1's in any preceding rows.

  Examples:

Reduced Row-Echelon Form:
  It is in row-echelon form.
  Each column containing a leading 1 has zeros in all its other entries.

  Examples:

Gauss-Jordan Elimination:

We use row operations to transform the augmented matrix into an equivalent matrix that is in reduced row-echelon form.

Example 8:   Solve the system $\begin{cases} 2x_1 + x_2 = 7 \\ 3x_1 + 2x_2 = 12 \end{cases}$.

Our goal: to use these row operations to derive an augmented matrix of the form $\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$.

Example 9:   Solve the system using Gauss-Jordan elimination.

$$3y - z = -1$$
$$x + 5y - z = -4$$
$$-3x + 6y + 2z = 11$$

Example 10: Solve the system using Gauss-Jordan elimination.

$$2x + y = z + 1$$
$$2x = 1 + 3y - z$$
$$-x + y + z = 4$$

Example 11:   Solve the system using Gauss-Jordan elimination.

$$3x - y + 2z = 2$$
$$2x - 2y + 5z = 7$$

Example 12:   Solve the system using Gauss-Jordan elimination.

$$-4x + 8y = 2$$
$$x - 3y = 4$$
$$2x + 2y = -3$$