5.3: Linear Programming in Two Dimensions: Geometric Approach

Linear programming is a valuable tool used for decision-making. It is often used to optimize a process. For example, it could be used to maximize profit, minimize cost, minimize down-time, etc.

The *objective function* is the quantity which is to be maximized or minimized.

The *constraints* are other requirements that must be satisfied (these are generally inequalities).

The objective function and the constraints together give us a *mathematical model*.

> Corner Point Theorem:
>
> If the feasible region is bounded, then the objective function has both a maximum and a minimum value and each occurs at one or more corner points.
>
> If the feasible region is unbounded, the objective function may not have a maximum or minimum. But if a maximum or minimum value exists, it will occur at one or more corner points.

General process for linear programming:

Step 1: Summarize the data in a table.

Step 2: Form a mathematical model:
  a. Write down the decision variables.
  b. Write down the objective function,.
  c. Write the problem constraints.
  d. Write any nonnegative constraints (e.g. $x \geq 0$, $y \geq 0$).

Step 3: Graph the feasible region and determine the coordinates of all corner points.

Step 4: Make a table listing the value of the objective function at each corner point.

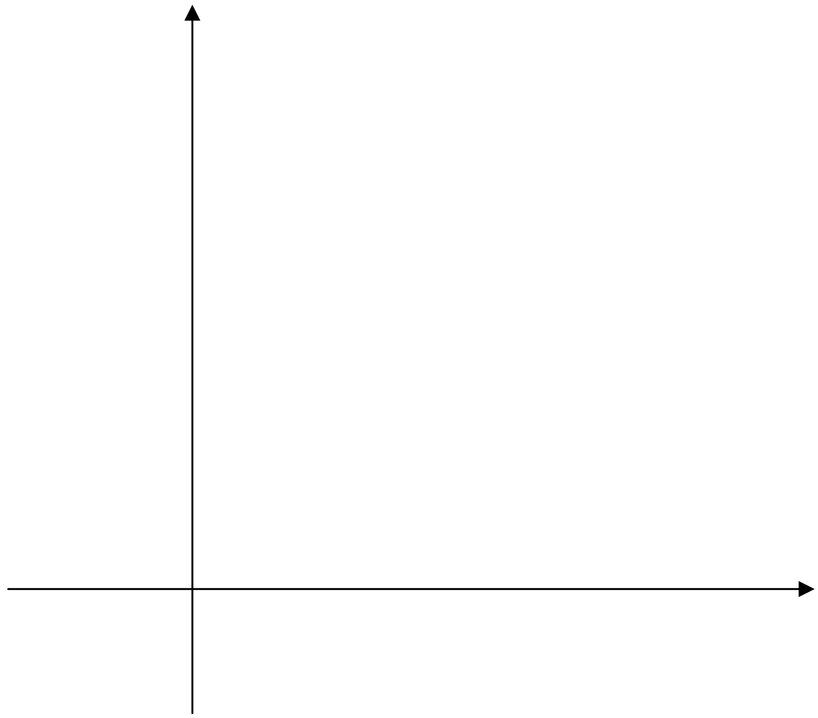
Step 5: Determine the optimal solution(s) from the table.

Step 6: Interpret the solution(s) in terms of the original problem.

Example 1: Solve the linear programming problem given by the following mathematical model.

Minimize $C = 100x_1 + 300x_2$

subject to
$$\begin{aligned} 2x_1 + x_2 &\geq 6 \\ 2x_1 + 4x_2 &\geq 12 \\ x_1 + 4x_2 &\geq 8 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

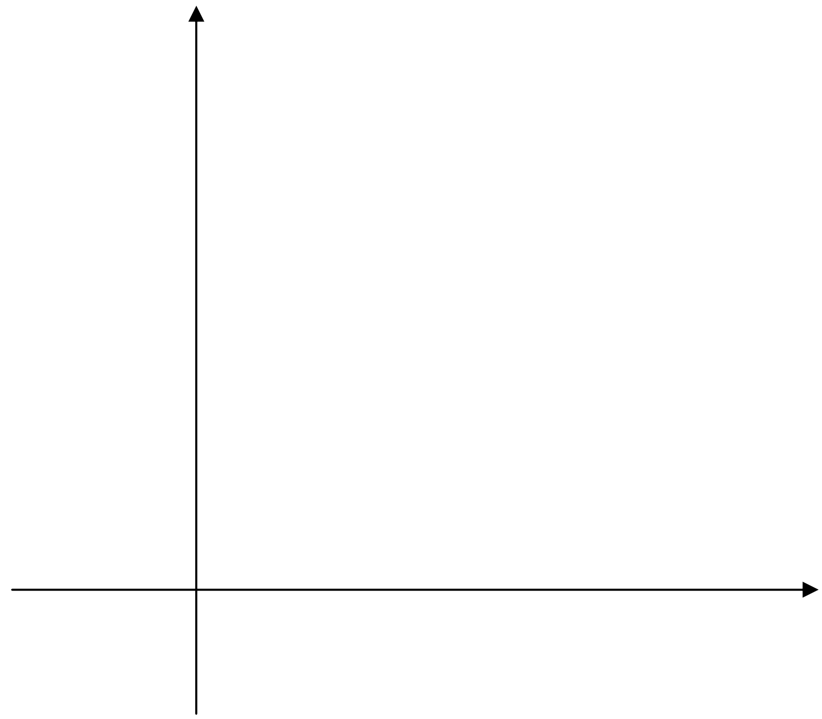


Example 2: Solve the linear programming problem given by the following mathematical model.

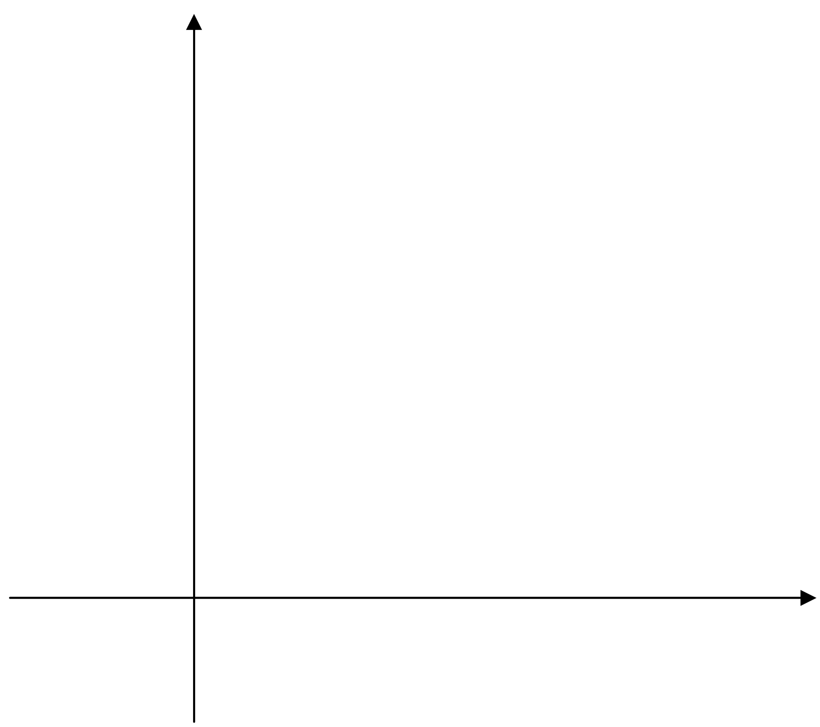
Minimize $z = 10x + 20y$

subject to $6x + 2y \geq 36$.

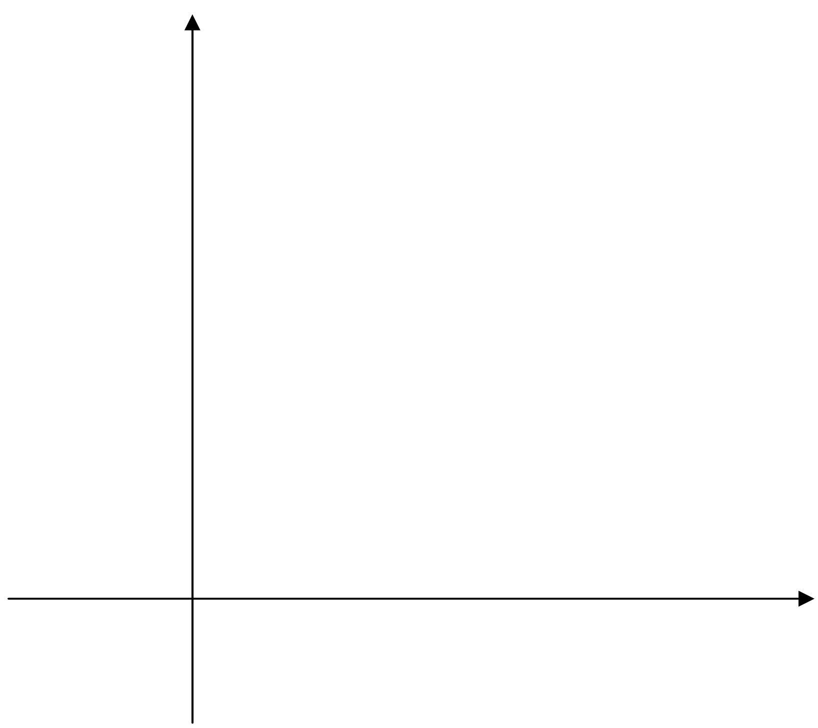
$$2x + 4y \geq 32$$
$$y \leq 20$$
$$x, y \geq 0$$



Example 3: A furniture company manufactures tables and chairs. There are two departments that work on these items: assembly and finishing. To complete a table requires 8 labor-hours in assembly and 2 labor-hours in finishing, while a chair requires 2 labor-hours in assembly and 1 labor-hour in finishing. The assembly department can work at most 400 labor-hours in one day and the finishing department can work at most 120 labor-hours. Each table earns the company $90 profit and each chair earns $25 profit. Assuming that the company will sell each item that it makes, how many tables and chairs should be made to maximize profit?



<u>Example 4:</u> A farmer can buy two types of fertilizer: *Grow'em Quick* and *Green and Tall*. Each cubic yard of *Grow'em Quick* contains 20 pounds of phosphoric acid, 30 pounds of nitrogen, and 5 pounds of potash. Each cubic yard of *Green and Tall* contains 10 pounds of phosphoric acid, 30 pounds of nitrogen, and 10 pounds of potash. The minimum monthly requirements are 460 pounds of phosphoric acid, 960 pounds of nitrogen, and 220 pounds of potash. If *Grow'em Quick* costs $44 per cubic yard and *Green and Tall* costs $32 per cubic yard, how many cubic yards of each mix should the farmer blend to meet the monthly requirements at a minimal cost? What is this cost?



Example 5: A high school is planning to rent buses and vans for a class trip. Each bus can transport 40 students, requires 3 chaperones, and costs $1200 to rent. Each van can transport 8 students, requires 1 chaperone, and costs $100 to rent. At least 400 students must be accommodated and only 36 parents have volunteered to serve as chaperones. How many vehicles of each type should be rented in order to minimize cost? What is the minimal cost?