

8.3: Conditional Probability, Intersection, and Independence

Conditional probability:

Consider the probability that a house will be flooded during a given year. Would you expect this probability to be different if you only considered houses that were located in a 50-year flood plain?

Example 1: Draw a single card from a standard 52-card deck.

- a. What is the probability that you draw a jack?

$$P(J) = \frac{4}{52} = \boxed{\frac{1}{13}}$$

J = set of jacks

- b. New information.... Given that you drew a face card (K, Q, J), what is the probability that it is a jack?

Could think of new sample space: All K, Q, J

$$n(S) = 3(4) = 12$$

Prob of getting a Jack is $\frac{4}{12} = \boxed{\frac{1}{3}}$

J = jacks, F = face cards. Then $P(J|F) = \boxed{\frac{1}{3}}$

Notation: $P(A|B)$ denotes the probability of A given that B occurs.

Conditional probability definition:

The probability of A given that B occurs is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (P(B) \neq 0)$$

$A|B$

Read

"A given B"

- c. Use the conditional probability definition to determine the probability that a jack is drawn, given that the card is a face card.

S = set of 52 cards
 F = set of face cards
 J = set of jacks

$$J \cap F = \{J\heartsuit, J\spadesuit, J\clubsuit, J\diamondsuit\}$$

$$P(J \cap F) = \frac{4}{52}$$

$$P(F) = \frac{12}{52}$$

$$P(J|F) = \frac{P(J \cap F)}{P(F)} = \frac{4/52}{12/52} = \frac{4}{52} \cdot \frac{52}{12} \ll \frac{4}{12} = \boxed{\frac{1}{3}}$$

Probability of the intersection divided by probability of the "given".

Example 2: Draw a single card from a standard 52-card deck. What is the probability of drawing the ace of diamonds given that the card is red?

1st way: Think of sample space as the set of all red cards
 so prob is $\boxed{\frac{1}{26}}$

2nd way: Use the formula.

$$n(S) = 52$$

$$R = \text{set of red cards}, A = \{A \diamond\}$$

$$A \cap R = \{A \diamond\} \Rightarrow P(A \cap R) = 1/52$$

$$P(R) = \frac{26}{52} \Rightarrow P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{1/52}{26/52} = \boxed{\frac{1}{26}}$$

Example 3: When rolling a single die, what is the probability of rolling a prime given that the number rolled is even?

$$P = \text{set of primes} = \{2, 3, 5\}$$

$$S = \{1, 2, \dots, 6\}$$

$$E = \{2, 4, 6\}$$

$$P \cap E = \{2\}$$

$$\text{so } P(P \cap E) = \frac{1}{6}$$

$$P(E) = \frac{3}{6} \Rightarrow P(P|E) = \frac{P(P \cap E)}{P(E)} = \frac{1/6}{3/6} = \boxed{\frac{1}{3}}$$

Example 4: In a test conducted by the U.S. Army, it was found that of 1000 new recruits, 680 men and 320 women, 57 of the men and 3 of the women were red-green color-blind. Given that a recruit selected at random from this group is red-green color-blind, what is the probability that the recruit is a male?

$$S = \text{set of 1000 new recruits} \quad \begin{cases} P(M \cap C) = \frac{57}{1000} \\ P(C) = \frac{60}{1000} \end{cases}$$

$$M = \text{set of men}$$

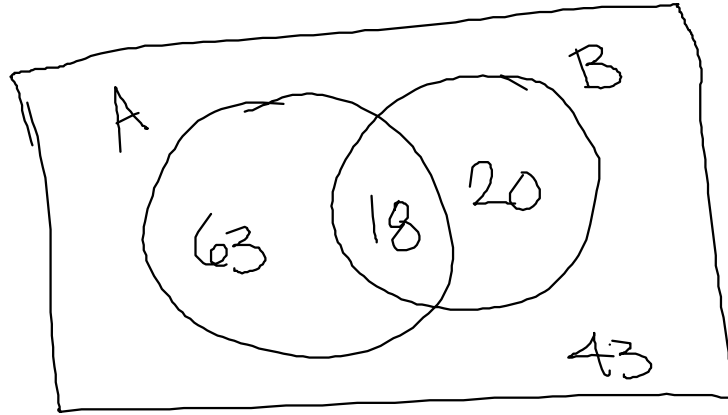
$$C = \text{set of color blind people}$$

$$P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{57/1000}{60/1000} = \boxed{\frac{57}{60}} \Rightarrow \boxed{0.95}$$

OR change the sample space to
 $S_2 = \text{the set of color-blind people}$
 $n(S_2) = 60$, so $P(M) = \boxed{\frac{57}{60}}$ $M = \text{men}$

Ex 4 $\frac{1}{2}$:

Consider this Venn diagram



Note

$$n(S) = 63 + 18 + 20 = 43$$

Find $P(B|A)$:

Method 1: Use conditional probability formula:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{18/43}{(63+18)/43}$$

$$= \frac{18}{63+18} \approx 0.22$$

Method 2:

Think of the sample space as just A
(so $n(S_2) = 63 + 18 = 81$)

$$\text{Then } P(B) = \frac{18}{81}$$

The product rule for intersections of events:Recall: The definition for conditional probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiplying both sides by $P(A)$: $P(B|A)P(A) = P(B \cap A)$

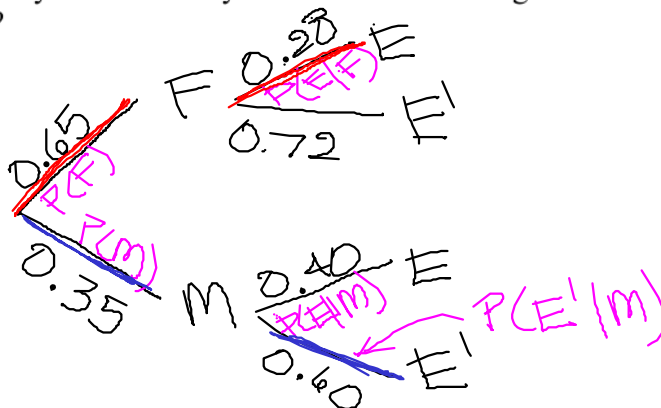
Product rule:For events A and B with nonzero probabilities

$$P(A \cap B) = P(B|A)P(A) = P(A)P(B|A)$$

Example 5: In a certain class, 65% of the students are female. 40% of the males and 28% of the females are engineering majors.

- a. What is the probability of a randomly selected student being female and an engineering major?

F = females
 $M = F' =$ males
 E = engineering majors

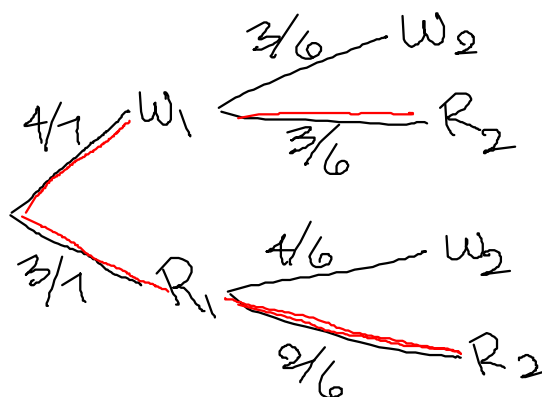
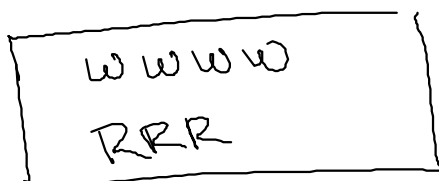


$$P(F \cap E) = P(F)P(E|F) = 0.65(0.28) = \boxed{0.182}$$

- b. What is the probability of a randomly selected student being male and a non-engineering major?

$$P(M \cap E') = 0.35(0.60) = \boxed{0.21}$$

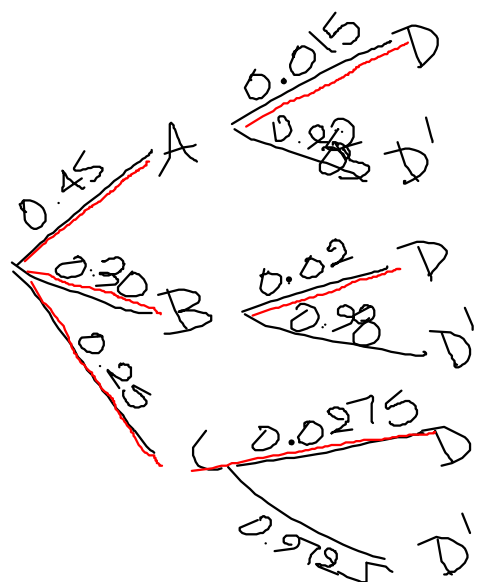
Example 6: A box contains 4 white marbles and 3 red marbles. Two marbles are drawn in succession without replacement. What is the probability of drawing a red marble on the second draw?



$$P(R_2) = \frac{4}{7} \left(\frac{3}{6} \right) + \frac{3}{7} \left(\frac{2}{6} \right) = \frac{12}{42} + \frac{6}{42} = \frac{18}{42} = \boxed{\frac{3}{7}}$$

Example 7: A certain type of camera is manufactured in three locations. Plants A, B, and C supply 45%, 30%, and 25%, respectively, of the cameras. The quality-control department of the company has determined that 1.5% of the cameras produced by plant A, 2% of the cameras produced by plant B and 2.75% of the cameras produced by plant C are defective. What is the probability that a randomly selected camera is defective?

D = the randomly chosen camera is defective



$$P(D) = 0.45(0.015) + 0.30(0.02) + 0.25(0.0275)$$

$$= \boxed{0.019625}$$

Example 8: A box contains eight 9-volt transistor batteries, of which two are known to be defective. The batteries are selected one at a time without replacement and tested until a nondefective one is found. What is the probability that the number of batteries tested is

- a. one?
- b. two?
- c. three?

Independence of events:

Example 9: (Compare with Example 6) A box contains 4 white marbles and 3 red marbles. Two marbles are drawn in succession, this time replacing the first before drawing the second.

- a. What is the probability of drawing a red marble on the second draw?
- b. What is the probability of drawing a red marble on the second draw given that a red marble was drawn on the first draw?

Two events are said to be *independent* if the outcome of one does not affect the outcome of the other. If they are not independent, then they are said to be *dependent*.

Independent Events:

Events A and B are independent events if and only if:

- $P(A|B) = P(A)$ or, equivalently,
- $P(B|A) = P(B)$ or, equivalently,
- $P(A \cap B) = P(A)P(B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A|B)P(B) = P(B \cap A)$$

If A & B are independent, $P(A|B) = P(A)$
 So $P(A)P(B) = P(B \cap A)$

Example 10: Draw a single card from a standard deck. Show whether the following pairs of events are independent or dependent.

- Drawing a heart and drawing a face card.
- Drawing a king and drawing a queen.

H: Heart

F: Face cards

$$\begin{aligned} a) P(H) &= \frac{13}{52} = \frac{1}{4} \\ P(F) &= \frac{12}{52} = \frac{3}{13} \end{aligned} \quad \left. \begin{aligned} &P(H)P(F) = \frac{1}{4} \left(\frac{3}{13} \right) = \frac{3}{52} \\ &P(H \cap F) = \frac{3}{52} \end{aligned} \right\} =$$

$P(H \cap F) = P(H)P(F)$,
 So H, F are independent.

Note $H \cap F = \{J, Q, K \text{ of hearts}\}$

$$b) P(K) = \frac{4}{52} = \frac{1}{13}$$

$$P(Q) = \frac{4}{52} = \frac{1}{13}$$

$$P(K \cap Q) = 0, \text{ because}$$

$$K \cap Q = \emptyset$$

$$\text{Is } P(K)P(Q) = P(K \cap Q)?$$

$$\text{No, } P(K)P(Q) = \frac{1}{13} \left(\frac{1}{13} \right) = \frac{1}{169} \neq 0$$

So K and Q are dependent events

Example 11: A survey conducted found that of 2000 women, 680 were heavy smokers and 50 had emphysema. Of those who had emphysema, 42 were also heavy smokers. Using this data, determine whether the events “being a heavy smoker” and “having emphysema” were independent events.

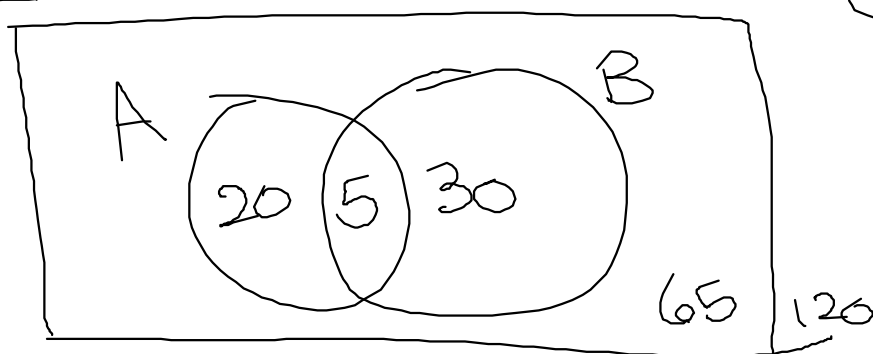
Independence of more than two events:

If $E_1, E_2, \dots, E_n$ are independent, then

$$P(E_1 \cap E_2 \cap \dots \cap E_n) = P(E_1)P(E_2) \dots P(E_n).$$

Example 12: A certain loudspeaker system has four components: a woofer, a midrange, a tweeter, and an electrical crossover. It has been determined that on the average 1% of the woofers, 0.8% of the midranges, 0.5% of the tweeters, and 1.5% of the crossovers are defective. Determine the probability that a randomly chosen loudspeaker is not defective. Assume that the defects in the different types of components are unrelated.

Example



②

Are A and B independent?

$$P(A) = \frac{25}{120}$$

$$P(B) = \frac{35}{120}$$

$$P(A)P(B) = 0.061$$

$$P(A \cap B) = \frac{5}{120} \approx 0.042$$

So A and B are dependent

g) "Fix" the sample space so A and B are independent. (move some elements from A to/from B)