3.2: Compound Interest

If at the end of a payment period, the interest due is reinvested at the same rate, then the interest as well as the principal will earn interest. This is called *compound interest*. The interest is paid into the account at the end of each compounding period.

Example 1: Suppose you invest \$1000 compounded quarterly at an annual interest rate of 8%. How much money will you have after one year?

quarterly $\Rightarrow$ 4 times per year

$r = 0.08$

1st quarter: $P = \$1000,\ r = 0.08,\ t = \frac{1}{4} = 0.25$

$A = P + Prt = P(1+rt) = \$1000\left(1 + 0.08\left(\frac{1}{4}\right)\right) = \$1000(1.02) = \$1020$

2nd quarter: $P = \$1000(1.02) = \$1020,\ r = 0.08,\ t = \frac{1}{4}$

$A = P(1+rt) = \$1020\,(\underbrace{1 + 0.08(\tfrac{1}{4})}_{1.02}) = 1000(1.02)(1.02) = \$1040.40$

3rd Qtr: $A = \$1000(1.02)(1.02)(1.02)$

4th Qtr: $\$1000(1.02)^4$

Compound Interest:

$$A = P(1+i)^n$$
$$= P\left(1 + \frac{r}{m}\right)^n$$
$$= P\left(1 + \frac{r}{m}\right)^{mt}$$

$A = P\left(1 + \frac{r}{m}\right)^n$

$A = P\left(1 + \frac{r}{m}\right)^{mt}$ because $n = mt$

where

$i = \frac{r}{m}$ is the interest rate per compounding period

$r =$ annual interest rate

$m =$ number of compounding periods per year

$n =$ total number of compounding periods

$P =$ principal (present value)

$A =$ amount (future value) at the end of n compounding periods.

Example 2: What is the future value of $1000 after 8 years at 6% compounded monthly? $\rightarrow m=12$

$$A = P(1+i)^n$$

$$A = \$1000\left(1+\frac{0.06}{12}\right)^{96}$$

$$= \boxed{\$1614.^{14}}$$

$i = \frac{r}{m} = \frac{0.06}{12}$

$n = mt = 12(8) = 96$

$P = \$1000$

$A = ?$

Example 3: How much should I invest now at 4% interest compounded monthly in order to have $10,000 in 6 years?

$$A = P\left(1+\frac{r}{m}\right)^n$$

$$10\,000 = P\left(1+\frac{0.04}{12}\right)^{72}$$

$$P = \$7869.418774$$

$$\boxed{P = \$7869.42}$$

$r = 0.04$

$m = 12$

$P = ?$

$A = \$10\,000$

$n = mt = 12(6) = 72$

Example 4: You decide to invest some money so that you will have $1,000,000 on your 75th birthday. At 8% compounded quarterly, how much should you invest on your 25th birthday?

$$A = P\left(1+\frac{r}{m}\right)^n$$

$$1\,000\,000 = P\left(1+\frac{0.08}{4}\right)^{200}$$

$$\boxed{P = \$19053.^{10}}$$

$t = 50$

$m = 4$

$n = mt = 4(50) = 200$

$A = \$1\,000\,000$

$P = ?$

Recall: logarithms: $2^3=8 \Rightarrow \log_2 8 = 3$

Example 5: How long will it take \$5,000 to grow to \$7,000 if it is invested at 8% compounded monthly?

$A = P\left(1+\frac{r}{m}\right)^n$

$\$7000 = \$5000\left(1+\frac{0.08}{12}\right)^{12t}$

Divide both sides by 1000:

$\frac{7000}{5000} = \left(1+\frac{0.08}{12}\right)^{12t}$

$\frac{7}{5} = \left(1+\frac{0.08}{12}\right)^{12t}$

Take ln (natural log) of both sides:

$\ln(7/5) = \ln\left(1+\frac{0.08}{12}\right)^{12t}$

$\ln\left(\frac{7}{5}\right) = 12t\ln\left(1+\frac{0.08}{12}\right)$

Property of logs

$\log_b x^r = r\log_b x$

$\ln = \log_e$

$\ln x^r = r\ln x$

$\frac{r}{m} = \frac{0.08}{12}$

$n = mt = 12t$

$t = ?$

$A = \$7000$

$P = \$5000$

$\frac{\ln(7/5)}{12\ln\left(1+\frac{0.08}{12}\right)} = t$

$t \approx 4.21999$

$t \approx \boxed{4.22 \text{ years}}$

Example 6: How long will it take money to double if it is invested at 7.5% compounded monthly?

You could use $\$1000 = P$, $\$2000 = A$

or, let $A = 2P$

$n = mt = 12t$

$A = P\left(1+\frac{r}{m}\right)^n$

$2P = P\left(1+\frac{0.075}{12}\right)^{12t}$

$\frac{2\cancel{P}}{\cancel{P}} = \left(1+\frac{0.075}{12}\right)^{12t}$

$2 = \left(1+\frac{0.075}{12}\right)^{12t}$

$\ln(2) = \ln\left(1+\frac{0.075}{12}\right)^{12t}$

$\ln(2) = 12t\ln\left(1+\frac{0.075}{12}\right)$

$\boxed{t \approx 9.27 \text{ years}}$

Continuous compound interest:

In calculus, a fundamental topic is the *limit*, or limiting value of a function. If we allow the number of compounding periods per year to increase toward infinity, the amount A approaches the limiting value $A = Pe^{rt}$. The number e is a constant, $e \approx 2.71828$. The number e is irrational—it cannot be written as a fraction of integers, or as a decimal that ends or repeats.

e can be defined as the limiting value of $\left(1+\frac{1}{x}\right)^x$ as x approaches ∞.

Start with the compound interest formula:

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

Substitute $x = \frac{m}{r}$ and then rearrange/simplify:

$$A = P\left[\left(1+\frac{1}{x}\right)^x\right]^{rt}$$

As $x \to \infty$, $\left(1+\frac{1}{x}\right)^x \to e$. This gives us the formula for continuous compound interest.

Continuous Compound Interest: If principal P is compounded continuously at the annual interest rate r, then the amount at the end of t years is $A = Pe^{rt}$. $\quad A = Pe^{rt}$

Example 7: How much must be invested now to have \$60,000 available in 10 years, if it is invested at 7% compounded (a) monthly? (b) continuously?

(a) $A = P\left(1+\frac{r}{m}\right)^n$

$60\,000 = P\left(1+\frac{0.07}{12}\right)^{120}$

$\boxed{P = \$29\,855.78}$

$A = \$60\,000$

$P = \$?$

$n = mt = 12(10) = 120$

$r = 0.07$

$m = 12$

(b) $A = Pe^{rt}$

$60\,000 = Pe^{0.07(10)}$

$60\,000 = P(2.01375) \Rightarrow \boxed{\$\,29\,795.12}$

$t = 10$

Example 8: How long will it take \$5,000 to grow to \$7,000 if it is invested at 8% compounded continuously?

$A = Pe^{rt}$

$7000 = 5000e^{0.08t}$

$\frac{7000}{5000} = e^{0.08t}$

$\frac{7}{5} = e^{0.08t}$

$\ln\left(\frac{7}{5}\right) = \ln\left(e^{0.08t}\right)$

$\ln\left(\frac{7}{5}\right) = 0.08t \Rightarrow t = 4.2059$

$\boxed{t \approx 4.21 \text{ years}}$

Effective rates:

The effective rate, sometimes called the *annual percentage yield*, converts a compound interest rate to an equivalent simple interest rate. This allows us to compare interest rates which have different compounding periods.

Annual Percentage Yield (APY):

The annual percentage yield (APY), or effective rate, is given by

$$APY = r_e = \left(1+\frac{r}{m}\right)^m - 1,$$

where

$r =$ annual interest rate

$m =$ number of compounding periods per year.

For interest compounded continuously, the APY is

$$APY = r_e = e^r - 1.$$

Example 9: What is the annual percentage yield for money invested at 6% compounded quarterly?

$$APY = r_e = \left(1+\frac{r}{m}\right)^m - 1$$

$r = 0.06$
$m = 4$

$$= \left(1+\frac{0.06}{4}\right)^4 - 1$$

$$= 1.06136 - 1 = 0.06136$$

$$\Rightarrow \boxed{6.136\%}$$

Example 10: Which investment is better, Note A at 9% compounded monthly or Note B at 9.2% compounded semiannually?

Calculate the APY for each. The better investment will have the higher APY, or r_e (effective rate).

See video if needed.