3.2: Compound Interest

If at the end of a payment period, the interest due is reinvested at the same rate, then the interest as well as the principal will earn interest. This is called *compound interest*. The interest is paid into the account at the end of each compounding period.

Example 1: Suppose you invest \$1000 compounded quarterly at an annual interest rate of 8%. How much money will you have after one year?

Compound Interest:

$$A = P(1+i)^n$$
$$= P\left(1+\frac{r}{m}\right)^n$$
$$= P\left(1+\frac{r}{m}\right)^{mt}$$

where

$i = \frac{r}{m}$ is the interest rate per compounding period

$r =$ annual interest rate

$m =$ number of compounding periods per year

$n =$ total number of compounding periods

$P =$ principal (present value)

$A =$ amount (future value) at the end of n compounding periods.

<u>Example 2:</u> What is the future value of $1000 after 8 years at 6% compounded monthly?

<u>Example 3:</u> How much should I invest now at 4% interest compounded monthly in order to have $10,000 in 6 years?

<u>Example 4:</u> You decide to invest some money so that you will have $1,000,000 on your 75th birthday. At 8% compounded quarterly, how much should you invest on your 25^{th} birthday?

Example 5: How long will it take \$5,000 to grow to \$7,000 if it is invested at 8% compounded monthly?

Example 6: How long will it take money to double if it is invested at 7.5% compounded monthly?

Continuous compound interest:

In calculus, a fundamental topic is the *limit*, or limiting value of a function. If we allow the number of compounding periods per year to increase toward infinity, the amount *A* approaches the limiting value $A = Pe^{rt}$. The number *e* is a constant, $e \approx 2.71828$. The number *e* is irrational—it cannot be written as a fraction of integers, or as a decimal that ends or repeats.

e can be defined as the limiting value of $\left(1+\frac{1}{x}\right)^x$ as *x* approaches ∞.

Start with the compound interest formula:

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

Substitute $x = \frac{m}{r}$ and then rearrange/simplify:

$$A = P\left[\left(1+\frac{1}{x}\right)^x\right]^{rt}$$

As $x \to \infty$, $\left(1+\frac{1}{x}\right)^x \to e$. This gives us the formula for continuous compound interest.

<u>Continuous Compound Interest</u>:

If principal *P* is compounded continuously at the annual interest rate *r*, then the amount at the end of *t* years is

$$A = Pe^{rt} .$$

<u>Example 7:</u> How much must be invested now to have \$60,000 available in 10 years, if it is invested at 7% compounded (a) monthly? (b) continuously?

<u>Example 8:</u> How long will it take $5,000 to grow to $7,000 if it is invested at 8% compounded continuously?

Effective rates:

The effective rate, sometimes called the *annual percentage yield*, converts a compound interest rate to an equivalent simple interest rate. This allows us to compare interest rates which have different compounding periods.

> Annual Percentage Yield (APY):
>
> The annual percentage yield (APY), or effective rate, is given by
>
> $$APY = r_e = \left(1 + \frac{r}{m}\right)^m - 1,$$
>
> where
>
> $r =$ annual interest rate
>
> $m =$ number of compounding periods per year.
>
> For interest compounded continuously, the APY is
>
> $$APY = r_e = e^r - 1.$$

Example 9: What is the annual percentage yield for money invested at 6% compounded quarterly?

Example 10: Which investment is better, Note A at 9% compounded monthly or Note B at 9.2% compounded semiannually?