4.5: Inverse of a Square Matrix

Identity matrix for multiplication:

For real numbers, 1 is the identity for multiplication. $1a = a \cdot 1 = a$ for all real numbers a

Is there an identity for matrix multiplication? i.e. is there a matrix I such that $MI = IM = M$?

However, for square matrices, there is such an identity.

For $n \times n$ matrices, I is the matrix with 1 on the principal diagonal and zeros elsewhere.

Examples of Identity Matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

2x2   3x3   4x4

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

Inverses:

Every real number except 0 has a multiplicative inverse.

Multiplicative inverse of $\frac{2}{3}$ is $\frac{3}{2}$

because $\frac{2}{3} \cdot \frac{3}{2} = 1$ ← multiplicative identity

Mult. inverse of -4 is $-\frac{1}{4}$

because $-\frac{4}{1}\left(-\frac{1}{4}\right) = \frac{4}{4} = 1$ = multiplicative identity

The Inverse of a Square Matrix:

Let M be an $n \times n$ square matrix and I be the $n \times n$ identity matrix. If there exists a matrix M^{-1} such that $M^{-1}M = MM^{-1} = I$, then M^{-1} is the inverse of M.

Note: $\left(\frac{2}{3}\right)^{-1} = \frac{2^{-1}}{3^{-1}} = \frac{3}{2}$

$(-4)^{-1} = \frac{1}{-4} = -\frac{1}{4}$

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Example 1: Verify that $\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$ are inverses of one another.

$$\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

R1C1: $5\left(\frac{1}{2}\right) + 3\left(-\frac{1}{2}\right) = \frac{5}{2} - \frac{3}{2} = \frac{2}{2} = 1$

R1C2: $5\left(-\frac{3}{2}\right) + 3\left(\frac{5}{2}\right) = -\frac{15}{2} + \frac{15}{2} = 0$

R2C2: $1\left(-\frac{3}{2}\right) + 1\left(\frac{5}{2}\right) = -\frac{3}{5} + \frac{5}{2} = \frac{2}{2} = 1$

R2C1: $1\left(\frac{1}{2}\right) + 1\left(-\frac{1}{2}\right) = 0$

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

R1C1: $5\left(\frac{1}{2}\right) + 1\left(-\frac{3}{2}\right) = \frac{5}{2} - \frac{3}{2} = \frac{2}{2} = 1$

So they are inverses.

How to find the inverse:

- To find the inverse of a matrix M, start by creating an augmented matrix $[M|I]$ by placing the appropriate-sized identity matrix to the right of the vertical line.
- Then row-reduce the augmented matrix until the identity matrix appears to the left of the vertical line. Then M^{-1} is to the right of the vertical line. In other words, row-reduce your augmented matrix until it looks like $[I|M^{-1}]$.
- If a zero row appears to the left of the vertical line, then M^{-1} does not exist.

Example 2: Find the inverse of $M = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, if it exists.

Goal: $\left[\begin{array}{cc|cc} 1 & 0 & * & * \\ 0 & 1 & * & * \end{array}\right]$

want 1

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right] \underset{\sim}{-2R_1 + R_2 \rightarrow R_2} \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

want 0 want 1

want 0

$$\underset{\sim}{-1R_2 \rightarrow R_2} \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right] \underset{\sim}{-2R_2 + R_1 \rightarrow R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

M^{-1}

$$\boxed{M^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}}$$

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Example 3: Find the inverse of $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, if it exists.

want 1

want 0

want 0

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -1R_1 + R_2 \to R_2 \\ \sim \\ 1R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

want 0

want 1

want 0

want 0s

$$\begin{array}{c} -1R_2 + R_1 \to R_1 \\ \sim \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

want 1

$$\begin{array}{c} 1R_3 + R_1 \to R_1 \\ \sim \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\boxed{M^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$$

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Example 4: Find the inverse of $M = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$, if it exists.

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Example 5: Find the inverse of $M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$, if it exists.

want 1

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right] \underset{\sim}{\xrightarrow{\frac{1}{2}R_1 \to R_1}} \left[\begin{array}{cccc} 1 & 1 & 1/2 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right]$$

want 0

$$\underset{\sim}{1R_1 + R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 \end{array}\right]$$

want 1. This is impossible!

$\boxed{M^{-1} \text{ does not exist}}$

M is singular. (does not have an inverse)

Shortcut for 2×2 matrices:

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the *determinant* of M is $D = ad - bc$. If $D \neq 0$, then M^{-1} exists and is given by

"trade places on the principal diagonal, change signs on the other diagonal"

$$M^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

bc ad

Example 6: Find the inverse of $M = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}$, if it exists.

determinant:

$$D = -5(4) - (-3)(6) = -20 + 18 = -2$$

$$M^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & -3/2 \\ 3 & 5/2 \end{bmatrix}}$$

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Example 7: Find the inverse of $M = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$, if it exists.

$D = 2(6) - 4(3) = 12 - 12 = 0$

$\boxed{M^{-1} \text{ does not exist}}$

Example 8: Find the inverse of $M = \begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix}$, if it exists.

Determinant: $D = 3(8) - 5(2) = 24 - 10 = 14$

$$M^{-1} = \frac{1}{14}\begin{bmatrix} 8 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 8/14 & -5/14 \\ -2/14 & 3/14 \end{bmatrix} = \boxed{\begin{bmatrix} 4/7 & -5/14 \\ -1/7 & 3/14 \end{bmatrix}}$$