4.5: Inverse of a Square Matrix

Identity matrix for multiplication:

For real numbers, 1 is the identity for multiplication.

Is there an identity for matrix multiplication? i.e. is there a matrix I such that $MI = IM = M$?

However, for square matrices, there is such an identity.

For $n \times n$ matrices, I is the matrix with 1 on the principal diagonal and zeros elsewhere.

Examples of Identity Matrices:

Inverses:

Every real number except 0 has a multiplicative inverse.

> The Inverse of a Square Matrix:
>
> Let M be an $n \times n$ square matrix and I be the $n \times n$ identity matrix. If there exists a matrix M^{-1} such that $M^{-1}M = MM^{-1} = I$, then M^{-1} is the inverse of M.

Example 1: Verify that $\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$ are inverses of one another.

How to find the inverse:

- To find the inverse of a matrix *M*, start by creating an augmented matrix $\left[M \middle| I \right]$ by placing the appropriate-sized identity matrix to the right of the vertical line.
- Then row-reduce the augmented matrix until the identity matrix appears to the left of the vertical line. Then M^{-1} is to the right of the vertical line. In other words, row-reduce your augmented matrix until it looks like $\left[I \middle| M^{-1} \right]$.
- If a zero row appears to the left of the vertical line, then M^{-1} does not exist.

Example 2: Find the inverse of $M = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, if it exists.

Example 3: Find the inverse of $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, if it exists.

Example 4: Find the inverse of $M = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$, if it exists.

Example 5: Find the inverse of $M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$, if it exists.

Shortcut for 2×2 matrices:

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the *determinant* of *M* is $D = ad - bc$. If $D \neq 0$, then M^{-1} exists and is given by

$$M^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 6: Find the inverse of $M = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}$, if it exists.

Example 7: Find the inverse of $M = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$, if it exists.

Example 8: Find the inverse of $M = \begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix}$, if it exists.