6.3: The Dual Problem: Minimization with ≥ Problem Constraints

The simplex method can be modified to solve minimization problems.

The transpose of a matrix:

The transpose of a matrix A is called A^T and is formed by interchanging the rows and columns of A.

Example 1: Find the transpose of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

Transpose of A is $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

The dual problem:

Every minimization problem with ≥ constraints can be associated with a maximization problem with ≤ constraints. This maximization problem is called the *dual problem*.

Example 1: Minimize $C = 2y_1 + y_2$

Subject to
$$y_1 + y_2 \geq 8$$
$$y_1 + 2y_2 \geq 4$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$

This is a (standard minimization problem: constraints all of the form $\geq C$.)

First, we create a matrix A using the constraints and the objective function, with the objective function on the bottom row:

$$A = \begin{array}{ccc} y_1 & y_2 & RHS \end{array}$$
$$A = \begin{bmatrix} 1 & 1 & 8 \\ 1 & 2 & 4 \\ 2 & 1 & * \end{bmatrix}$$

Last row is from $C = 2y_1 + y_2$
$2y_1 + y_2 = C$

Next, form the transpose A^T:

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 8 & 4 & * \end{bmatrix}$$

From the transpose, write a new linear programming problem with new variables:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 8 & 4 & * \end{bmatrix} \Rightarrow \begin{aligned} x_1 + x_2 &\le 2 \\ x_1 + 2x_2 &\le 1 \\ 8x_1 + 4x_2 &= z \end{aligned}$$

The dual problem is:

$$\begin{aligned} \text{Maximize} \quad & z = 8x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 2 \\ & x_1 + 2x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Notice: This is a standard maximization problem.

> Theorem of Duality:
>
> The objective function w (C) of a minimizing linear programming problem takes on a minimum value if and only if the objective function z of the corresponding dual maximizing problem takes on a maximum value. The maximum value of z (in the dual) is equal to the minimum value of w. (C (in the original))

So, after forming the dual problem, use the simplex method to solve it.

- For slack variables, use the variables of the original minimization problem.
- When writing the solution, the values of the original variables are read from the bottom row.

$$\left.\begin{aligned} x_1 + x_2 &\le 2 \\ x_1 + 2x_2 &\le 1 \\ z &= 8x_1 + 4x_2 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 + x_2 + y_1 \qquad &= 2 \\ x_1 + 2x_2 \qquad + y_2 \qquad &= 1 \\ -8x_1 - 4x_2 \qquad\qquad + z &= 0 \end{aligned}$$

$$\begin{array}{cccccc} x_1 & x_2 & y_1 & y_2 & z & \text{RHS} \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -8 & -4 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \frac{2}{1} = 2 \\ \frac{1}{1} = 1 \text{ smallest quotient} \\ \end{array}$$

most negative

$-1R_2 + R_1 \to R_1$

$8R_2 + R_3 \to R_3$

$$\begin{array}{cccccc} x_1 & x_2 & y_1 & y_2 & z & \text{RHS} \\ \end{array}$$
$$\begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 12 & 0 & 8 & 1 & 8 \end{bmatrix}$$

y_1 in original y_2 in original

No more negatives in bottom row, so we're done pivoting.

Solution to the Dual Problem: max $z = 8$, $x_1 = 1$, $x_2 = 0$.

Solution to the original min problem: $C = 8$, $y_1 = 0$, $y_2 = 8$

Example 2: (Example from Section 5.3—plant food)

Minimize $C = 30x_1 + 35x_2$

Subject to
$$20x_1 + 10x_2 \geq 460$$
$$30x_1 + 30x_2 \geq 960$$
$$5x_1 + 10x_2 \geq 220$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

write matrix A:

$$\begin{bmatrix} 20 & 10 & 460 \\ 30 & 30 & 960 \\ 5 & 10 & 220 \\ 30 & 35 & * \end{bmatrix}$$

Transpose is $A^T = \begin{bmatrix} 20 & 30 & 5 & 30 \\ 10 & 30 & 10 & 35 \\ \hline 460 & 960 & 220 & * \end{bmatrix}$

The dual problem is:

Maximize $P = 460y_1 + 960y_2 + 220y_3$

subject to:
$$20y_1 + 30y_2 + 5y_3 \leq 30$$
$$10y_1 + 30y_2 + 10y_3 \leq 35$$
$$y_1, y_2, y_3 \geq 0$$

write system of equations:

$$\begin{aligned} 20y_1 + 30y_2 + 5y_3 + x_1 \quad\quad &= 30 \\ 10y_1 + 30y_2 + 10y_3 \quad + x_2 \quad &= 35 \\ -460y_1 - 960y_2 - 220y_3 \quad\quad + P &= 0 \end{aligned}$$

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Ex 2 cont'd: Set up Simplex tableau:

$$\begin{array}{ccccccc} y_1 & y_2 & y_3 & x_1 & x_2 & P & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 20 & 30 & 5 & 1 & 0 & 0 & 30 \\ 10 & 30 & 10 & 0 & 1 & 0 & 35 \\ -460 & -960 & -220 & 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \frac{30}{30} = 1 \leftarrow \text{smallest quotient} \\ \frac{35}{30} = \frac{7}{6} = 1\frac{1}{6} \\ \\ \end{array}$$

most negative

(From online pivot tool)

∼

y_1	y_2	y_3	x_1	x_2	P	RHS
x1	x2	x3	x4	x5	x6	x7
2/3	1	1/6	1/30	0	0	1
-10	0	5	-1	1	0	5
180	0	-60	32	0	1	960

$\frac{1}{1/6} = 6$

$\frac{5}{5} = 1 \leftarrow$ smallest quotient

most negative

∼

y_1	y_2	y_3	x_1	x_2	P	RHS
x1	x2	x3	x4	x5	x6	x7
1	1	0	1/15	-1/30	0	5/6
-2	0	1	-1/5	1/5	0	1
60	0	0	20	12	1	1020

read the values of x_1 and x_2 in original problem from the bottom row.

(so $x_1 = 20$ and $x_2 = 12$ in original prob. But note that in the dual problem, this tableau has $x_1 = x_2 = 0$ (they're nonbasic))

No more negatives on bottom row, so we have arrived at the final tableau.

$\boxed{\begin{array}{l}\text{Solution to dual (maximization) problem: The maximum } P \text{ is } 1020, \\ \text{occurring when } y_1 = 0,\ y_2 = \frac{5}{6},\ y_3 = 1.\end{array}}$

$\boxed{\begin{array}{l}\text{Solution to original minimization problem} \\ \text{The minimum } C \text{ is } 1020, \text{ occurring when } x_1 = 20,\ x_2 = 12.\end{array}}$

Example 3: An oil company operates two refineries in a certain city. Refinery I has an output of 200, 100, and 100 barrels of low-, medium-, and high-grade oil per day, respectively. Refinery II has an output of 100, 200, and 600 barrels of low-, medium-, and high-grade oil per day, respectively. The company wishes to produce at least 1000, 1400, and 3000 barrels of low-, medium-, and high-grade oil to fill an order. If it costs $20,000/day to operate refinery I and $30,000/day to operate refinery II, determine how many days each refinery should be operated to meet the requirements of the order at minimum cost to the company.