4.2/4.3: Gausss – Jordan Elimination

Example: Solve the system using Gauss-Jordan elimination.

$$4x_1 - 2x_2 + 3x_3 = 3$$
$$3x_1 - x_2 - 2x_3 = -10$$
$$2x_1 + 4x_2 - x_3 = -1$$

Goal: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$

want 1

$$\left[\begin{array}{ccc|c} 4 & -2 & 3 & 3 \\ 3 & -1 & -2 & -10 \\ 2 & 4 & -1 & -1 \end{array}\right] \underset{\sim}{\tfrac{1}{4}R_1 \to R_1} \left[\begin{array}{cccc} 1 & -1/2 & 3/4 & 3/4 \\ 3 & -1 & -2 & -10 \\ 2 & 4 & -1 & -1 \end{array}\right]$$

want 0s

want 1 want 0

$$\begin{array}{c} -3R_1 + R_2 \to R_2 \\ \sim \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc} 1 & -1/2 & 3/4 & 3/4 \\ 0 & 1/2 & -17/4 & -49/4 \\ 0 & 5 & -5/2 & -5/2 \end{array}\right] \underset{\sim}{2R_2 \to R_2} \left[\begin{array}{cccc} 1 & -1/2 & 3/4 & 3/4 \\ 0 & 1 & -\frac{17}{2} & -\frac{49}{2} \\ 0 & 5 & -5/2 & -5/2 \end{array}\right]$$

want 0

$$\begin{array}{c} \frac{1}{2}R_2 + R_1 \to R_1 \\ \sim \\ -5R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{cccc} 1 & 0 & -7/2 & -23/2 \\ 0 & 1 & -17/2 & -49/2 \\ 0 & 0 & 40 & 120 \end{array}\right].$$

want 0s want 1

$$\begin{array}{c} \frac{1}{40}R_3 \to R_3 \\ \sim \end{array} \left[\begin{array}{cccc} 1 & 0 & -7/2 & -23/2 \\ 0 & 1 & -17/2 & -49/2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} \frac{7}{2}R_3 + R_1 \to R_1 \\ \sim \\ \frac{17}{2}R_3 + R_2 \to R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solution to system:
$$\boxed{\begin{aligned} x_1 &= -1 \\ x_2 &= 1 \\ x_3 &= 3 \end{aligned}}$$

4.2/3.2

Example: Solve the system using Gauss-Jordan elimination.

$$-4x_1 + 8x_2 + 10x_3 = -6$$
$$6x_1 - 12x_2 - 15x_3 = 9$$
$$-8x_1 + 14x_2 + 19x_3 = -8$$

want 1

$$\begin{bmatrix} -4 & 8 & 10 & -6 \\ 6 & -12 & -15 & 9 \\ -8 & 14 & 19 & -8 \end{bmatrix} \overset{-\frac{1}{4}R_1 \to R_1}{\sim} \begin{bmatrix} 1 & -2 & -5/2 & 3/2 \\ 6 & -12 & -15 & 9 \\ -8 & 14 & 19 & -8 \end{bmatrix}$$

want 0s

want 1   want 1

$$\begin{array}{c} -6R_1+R_2 \to R_2 \\ \sim \\ 8R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 1 & -2 & -5/2 & 3/2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 4 \end{bmatrix} \overset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & -2 & -5/2 & 3/2 \\ 0 & -2 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

want 0

$$\overset{-\frac{1}{2}R_2 \to R_2}{\sim} \begin{bmatrix} 1 & -2 & -5/2 & 3/2 \\ 0 & 1 & 1/2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{2R_2+R_1 \to R_1}{\sim} \begin{bmatrix} 1 & 0 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & 1/2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R_1: $x_1 - \frac{3}{2}x_3 = -\frac{5}{2}$

R_2: $x_2 + \frac{1}{2}x_3 = -2$

R_3: $0 = 0$

Let
$$\boxed{\begin{aligned} x_3 &= t \\ x_2 &= -\tfrac{1}{2}t - 2 \\ x_1 &= \tfrac{3}{2}t - \tfrac{5}{2} \end{aligned}}$$

4.4: Matrices-Basic Operations

We will learn how to add matrices and how to multiply a matrix by a number (scalar).

Equality:

Two matrices are *equal* if they are the same size *and* all the corresponding elements are equal.

Example 1:

$$A=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix},\ B=\begin{bmatrix}1&2\\3&4\end{bmatrix},\ C=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix},$$

$B=D$

$$D=\begin{bmatrix}1&2\\3&4\end{bmatrix},\ E=\begin{bmatrix}1&1\\1&1\end{bmatrix},\ F=\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}$$

Addition:

Matrices can be added only if they are the same size. In this case, we add the corresponding elements in the matrices.

Example 2:

$$\begin{bmatrix}1&3\\5&7\end{bmatrix}+\begin{bmatrix}1\\7\end{bmatrix}=$$ not defined (sizes don't match)

$$\begin{bmatrix}2&4\\4&2\end{bmatrix}+\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}3&6\\7&6\end{bmatrix}$$

$$\begin{bmatrix}-1\\2\\3\end{bmatrix}+\begin{bmatrix}4\\-5\\6\end{bmatrix}=\begin{bmatrix}3\\-3\\9\end{bmatrix}$$

Addition of numbers is associative and commutative. Addition of matrices is associative and commutative also.

- <u>Commutative</u>: $A+B=B+A$
- <u>Associative</u>: $(A+B)+C=A+(B+C)$

A *zero matrix* is a matrix with zero in all positions. The following are zero matrices of different sizes:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2×2 3×2

The *negative of a matrix A*, denoted $-A$, is the matrix with all elements that are the opposites of the corresponding elements in the matrix A.

Example 3:

$$A = \begin{bmatrix} 4 & -6 \\ 5 & 7 \end{bmatrix} \qquad -A = \begin{bmatrix} -4 & 6 \\ -5 & -7 \end{bmatrix}$$

Subtraction:

As with addition, subtraction can be performed only if matrices are the same size. The difference $A-B$ is defined to be $A+(-B)$. So to subtract, we just subtract the corresponding elements.

Example 4:

$$\begin{bmatrix} 1 & 2 & -6 \\ -3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 0 & -2 & -2 \\ 4 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -4 \\ -7 & -2 & 0 \end{bmatrix}$$

Multiplication of a matrix by a number:

The product of a number k and a matrix M, denoted by kM, is the matrix formed by multiplying each element of M by k. This is often called *scalar multiplication*.

Example 5:

$$4\begin{bmatrix} 1 & 3 \\ -2 & 7 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ -8 & 28 \\ 0 & -16 \end{bmatrix}$$

Product of a row matrix and a column matrix (in that order):

The product of a $1\times n$ row matrix A and an $n\times 1$ column matrix is the 1×1 matrix given by

$$\underset{1\times n}{AB=\begin{bmatrix} a_1 & a_2 \cdots a_n \end{bmatrix}}\underset{n\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}=\underset{1\times 1}{\begin{bmatrix} a_1b_1+a_2b_2+\cdots+a_nb_n \end{bmatrix}}.$$

Note: For this formula to hold, they must be in this order: row$\times$column. If they are in the other order (column$\times$row), you get a different result. We'll see one like this later.

Note: The number of elements in the row and column must be the same in order for the multiplication to be defined.

Example 6:

$$\underset{1\times 4}{\begin{bmatrix} 1 & -2 & 3 & -5 \end{bmatrix}}\underset{4\times 1}{\begin{bmatrix} -4 \\ 0 \\ 2 \\ -4 \end{bmatrix}}=\begin{bmatrix} 1(-4)-2(0)+3(2)-5(-4) \end{bmatrix}$$

$$-4+0+6+20=\boxed{22} \text{ or } \begin{bmatrix} 22 \end{bmatrix}$$

$$\begin{bmatrix} 21 & -32 & 19 \end{bmatrix}\begin{bmatrix} 11 \\ 23 \\ 13 \end{bmatrix}= 21(11)-32(23)+19(13)=\boxed{-258}$$

$$\underset{1\times 4}{\begin{bmatrix} 3 & -2 & 5 & 3 \end{bmatrix}}\underset{3\times 1}{\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}}= \text{not possible}$$

Matrix multiplication:

If A is an $m\times p$ and B is a $p\times n$ matrix, then the product of these is denoted AB and it is an $m\times n$ matrix.

$$(m\times \cancel{p})(\cancel{p}\times n)=(m\times n)$$

The entries in the matrix AB are formed as follows: the element in the ith row and jth column is the product of the ith row of A with the jth column of B.

Important Note: If the number of columns of A is not equal to the number of rows of B, the product AB is not defined! The matrices cannot be multiplied!!

Example 7:

$$A=\begin{bmatrix}2 & 1 & 3\\0 & -2 & -1\end{bmatrix},\ B=\begin{bmatrix}1 & 0\\4 & -2\\1 & 1\end{bmatrix} \Rightarrow \begin{bmatrix} & \end{bmatrix}_{2\times 2}$$

$2\times\cancel{3}$ match $\cancel{3}\times 2$

$$AB=\begin{bmatrix}2 & 1 & 3\\0 & -2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\4 & -2\\1 & 1\end{bmatrix}=\begin{bmatrix}\underset{R_1C_1}{9} & \_\\ \_ & \_\end{bmatrix}$$

$R_1C_1: 2(1)+1(4)+3(1)$
$2+4+3=9$

$$AB=\begin{bmatrix}2 & 1 & 3\\0 & -2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\4 & -2\\1 & 1\end{bmatrix}=\begin{bmatrix}9 & \\ \underset{R_2C_1}{-9} & \end{bmatrix}$$

$R_2C_1: 0(1)-2(4)-1(1)$
$0-8-1$
$=-9$

$$AB=\begin{bmatrix}2 & 1 & 3\\0 & -2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\4 & -2\\1 & 1\end{bmatrix}=\begin{bmatrix}9 & \underset{R_1C_2}{1}\\ -9 & \end{bmatrix}$$

$R_1C_2: 2(0)+1(-2)+3(1)$
$=0-2+3$
$=1$

$$AB=\begin{bmatrix}2 & 1 & 3\\0 & -2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\4 & -2\\1 & 1\end{bmatrix}=\begin{bmatrix}9 & 1\\ -9 & \underset{R_2C_2}{3}\end{bmatrix}$$

$R_2C_2: 0(0)-2(-2)-1(1)$
$0+4-1$
$=3$

Example 8:

$$\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\-1 & 4\end{bmatrix}=\begin{bmatrix}1 & 10\\5 & 8\end{bmatrix}$$

Example 9:

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}4 & 5 & 6\end{bmatrix}=\begin{bmatrix}4 & 5 & 6\\8 & 10 & 12\\12 & 15 & 18\end{bmatrix}$$

$3\times\cancel{1}$ $\cancel{1}\times3$ $\qquad$ 3×3

3×3

Example 10:

$$\begin{bmatrix}-1 & 3\\4 & 2\\-2 & 5\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\6 & 1 & -2\end{bmatrix}=\begin{bmatrix}17 & 1 & -9\\16 & 10 & 8\\28 & 1 & -16\end{bmatrix}$$

$3\times\cancel{2}$ $\cancel{2}\times3$ $\qquad$ 3×3

match

3×3

Example 11:

$$\begin{bmatrix}1 & 3 & 1\\2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}=$$

$\boxed{\text{Not possible}}$

2×3 $\quad$ 2×3

don't match!!

Example 12:

$$\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}\begin{bmatrix}2 & 6\\-1 & -3\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$2\times\cancel{2}$ $\cancel{2}\times2$ $\qquad$ 2×2

2×2

So Zero Product Property does not hold for matrix multiplication.

($AB=0$ does not imply that $A=0$ or $B=0$)

Example 13:

$$\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 20 & 40 \\ -10 & -20 \end{bmatrix}$$

compare this to example 12. Same two matrices, but multiplied in the opposite order.

Different order gave us a different result.

So, Commutative Property does not hold for matrix multiplication (in general, $AB \neq BA$)

Example 14:

$$\begin{bmatrix} 2 & -4 & 3 \\ -3 & 1 & 5 \\ 10 & -2 & 7 \end{bmatrix}^2$$

$$= \begin{bmatrix} 2 & -4 & 3 \\ -3 & 1 & 5 \\ 10 & -2 & 7 \end{bmatrix}\begin{bmatrix} 2 & -4 & 3 \\ -3 & 1 & 5 \\ 10 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 46 & -18 & 7 \\ 41 & 3 & 31 \\ 96 & -56 & 69 \end{bmatrix}$$