6.2: Trig Equations

6.2.1

Example: Solve over the interval $[0, 2\pi)$.

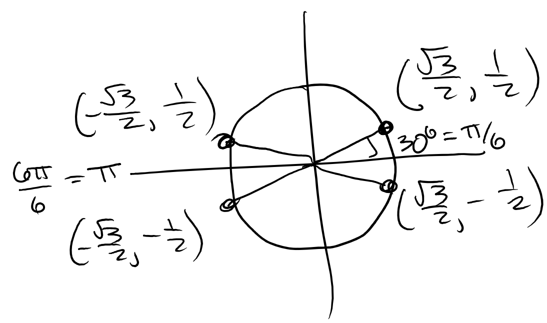
(Find all solutions x such that $0 \le x < 2\pi$)

$$2\sin x + 3 = 4$$

Isolate the trig function.

$$2\sin x + 3 = 4$$
$$-3 \quad -3$$
$$2\sin x = 1$$
$$\frac{2\sin x}{2} = \frac{1}{2}$$
$$\sin x = \frac{1}{2}$$

Think of $\sin\theta = \frac{1}{2}$



We want to find all angles x that make this true.

Quadrants 2 and 1, so

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Sol'n Set: $\boxed{\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}}$

Note: $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\boxed{6.2.2}$

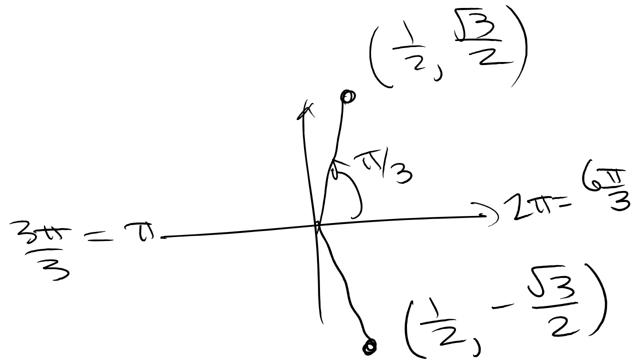
Ex: Solve on $[0, 2\pi)$.

$$2\sec x + 1 = \sec x + 3$$
$$-\sec x \qquad\qquad -\sec x$$

$$\sec x + 1 = 3$$
$$-1 \qquad -1$$

$$\sec x = 2$$

$$\cos x = \frac{1}{2}$$



Q I and Q IV

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Solution set: $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$

Part (b): Solve the same equation on $(-\infty, \infty)$.
(Find all solutions)

$\frac{\pi}{3} + n(2\pi)$, $\frac{5\pi}{3} + n(2\pi)$, where n is any integer

$\boxed{\frac{\pi}{3} + 2n\pi, \quad \frac{5\pi}{3} + 2n\pi, \text{ where } n \text{ is any integer}}$

Ex: Solve on (a) $[0, 2\pi)$
(b) $(-\infty, \infty)$ (Find all real number solutions)
(c) $[0^\circ, 360^\circ)$
(d) all degree-valued solutions

$$(\cot x - 1)(\sqrt{3}\cot x + 1) = 0$$

$\cot x - 1 = 0$

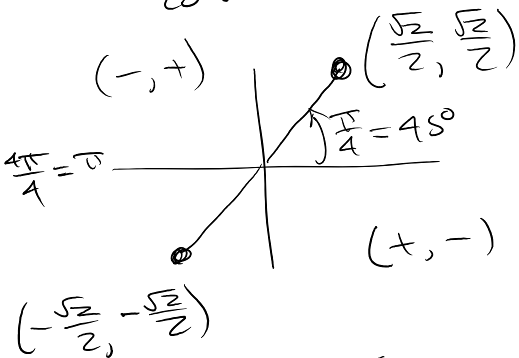
$\cot x = 1$

$\dfrac{\cos x}{\sin x} = 1$

or

$\tan x = \frac{1}{1} = 1$

$\dfrac{\sin x}{\cos x} = 1$



$x = \frac{\pi}{4}, \frac{5\pi}{4}$

OR

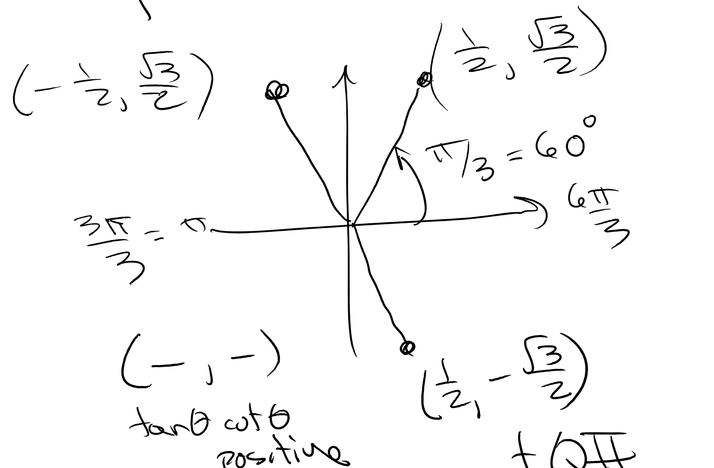
$\sqrt{3}\cot x + 1 = 0$

$\sqrt{3}\cot x = -1$

$\dfrac{\sqrt{3}\cot x}{\sqrt{3}} = \dfrac{-1}{\sqrt{3}}$

$\cot x = -\dfrac{1}{\sqrt{3}} \Rightarrow \dfrac{\cos x}{\sin x} = -\dfrac{1/2}{\sqrt{3}/2}$

$\tan x = -\dfrac{\sqrt{3}}{1} \Rightarrow \dfrac{\sin x}{\cos x} = -\dfrac{\sqrt{3}/2}{1/2}$



We want QII and QIV

$x = \frac{2\pi}{3}, \frac{5\pi}{3}$

See next page

(a) On $[0, 2\pi)$: Sol'n Set is $\left\{\frac{\pi}{4}, \frac{5\pi}{4}, \frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

(b) on $(-\infty, \infty)$: $\left\{\frac{\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi, \frac{2\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi\right\}$

where n is any integer

$(n \in \mathbb{Z})$

Shorthand:

$\mathbb{R}$: all real numbers

$\mathbb{Z}$ = set of all integers

(c) on $[0°, 360°)$: $\{45°, 225°, 120°, 300°\}$

(d) all degree solutions:

$\{45° + 360°n, 225° + 360°n, 120° + 360°n, 300° + 360°n\}$

6.2.5

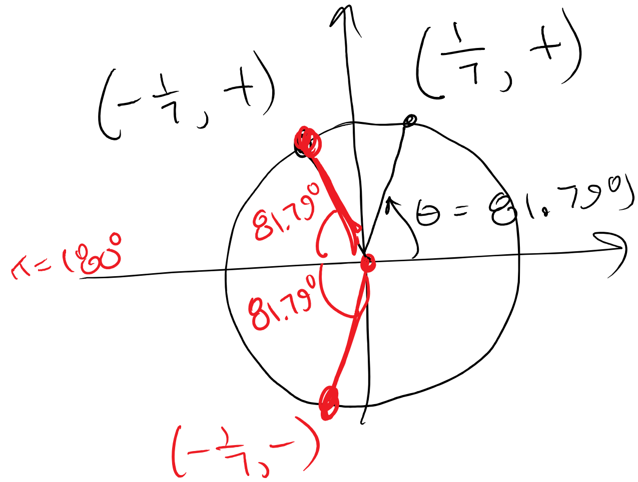
Ex: Find all degree-valued solutions.

$$-7\cos x + 1 = 2$$

$$-7\cos x = 1$$

$$\frac{-7\cos x}{-7} = \frac{1}{-7}$$

$$\cos x = -\frac{1}{7}$$



Find the reference angle:

$$\theta = \cos^{-1}\left(\frac{1}{7}\right) \approx 81.79°$$

We want $\cos x = -\frac{1}{7}$, so Q2 and Q3.

Q2 angle: $180° - 81.79° = 98.21°$

Q3 angle: $180° + 81.79° = 261.79°$

solving on $[0°, 180°)$, sol'n set is $\boxed{\{98.21°, 261.79°\}}$

If working in radian measure,

$$\theta = \cos^{-1}\left(\frac{1}{7}\right) \approx 1.427$$

Q2: $\pi - 1.427$

Q3: $\pi + 1.427$

6.2.6

Example: Find all solutions of

$$2\sin\theta - 1 = \csc\theta.$$

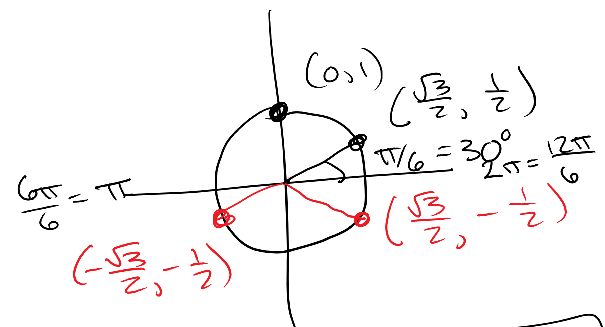
$$2\sin\theta - 1 = \frac{1}{\sin\theta}$$

$$2\sin^2\theta - \sin\theta = 1$$ [multiply both sides by $\sin\theta$]

$$2\sin^2\theta - \sin\theta - 1 = 0$$

$$(2\sin\theta + 1)(\sin\theta - 1) = 0$$

$2\sin\theta + 1 = 0$	$\sin\theta - 1 = 0$
$2\sin\theta = -1$	$\sin\theta = 1$
$\sin\theta = -\frac{1}{2}$	$\theta = \frac{\pi}{2}$



Q3 angle: $\frac{7\pi}{6}$

Q4 angle: $\frac{11\pi}{6}$

$$\boxed{\text{solutions on } [0, 2\pi) \text{ are } \left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}}$$

Now find all the solutions:

$$\boxed{\frac{\pi}{2} + 2\pi n,\ \frac{7\pi}{6} + 2\pi n,\ \frac{11\pi}{6} + 2\pi n, \text{ where } n \text{ is any integer}}$$

Ex: Find all solutions of $5 + 5\tan^2 x = 6\sec x$. 6.2.7

$$5 + 5\tan^2 x = 6 \sec x$$

Pythagorean Identity

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$$5 + 5(\sec^2 x - 1) = 6 \sec x$$

$$5 + 5\sec^2 x - 5 = 6 \sec x$$

$$5\sec^2 x = 6 \sec x$$

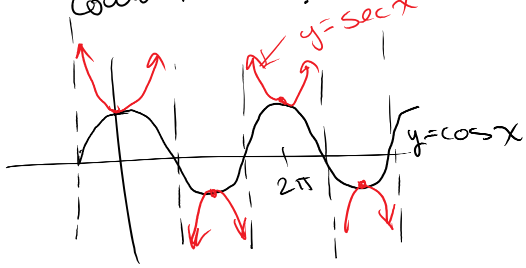
$$5\sec^2 x - 6\sec x = 0$$

$$\sec x(5\sec x - 6) = 0$$

$\sec x = 0$

$\cos x$ is undefined

Hmmm.... how could that be?



$y = \sec x$ never touches x-axis, so can't have

$y = \sec x = 0$.

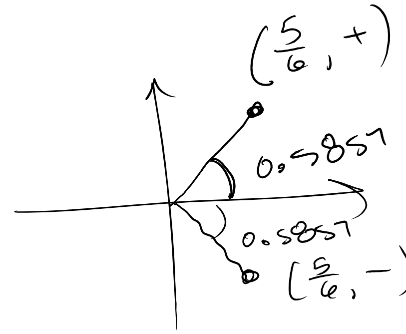
This factor doesn't contribute any solutions

$5\sec x - 6 = 0$

$5\sec x = 6$

$\sec x = \frac{6}{5}$

$\cos x = \frac{5}{6}$



$\cos^{-1}\left(\frac{5}{6}\right) \approx 0.5857$ (ref. angle)

Q1 angle: $x \approx 0.5857$

Q4 angle: $2\pi - \cos^{-1}\left(\frac{5}{6}\right)$

$\approx 2\pi - 0.5857$

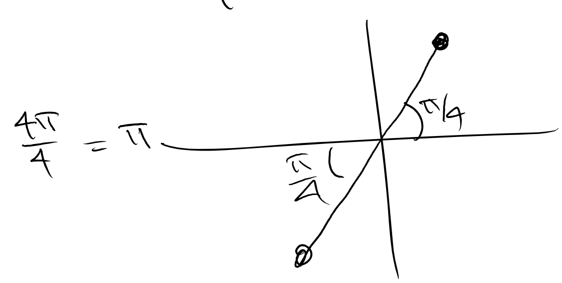
≈ 5.6975

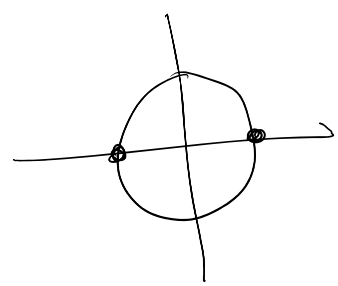
$\boxed{\text{Solutions in } [0, 2\pi) \text{ are } 0.5857,\ 5.6975}$

All solutions: $\boxed{0.5857 + 2\pi n,\ 5.6975 + 2\pi n,\ n \text{ any integer (or } n \in \mathbb{Z})}$

<u>Note:</u> If an equation has solutions

$\frac{\pi}{4} + 2n\pi$, $\frac{5\pi}{4} + 2n\pi$, $0 + 2n\pi$, $\pi + 2n\pi$



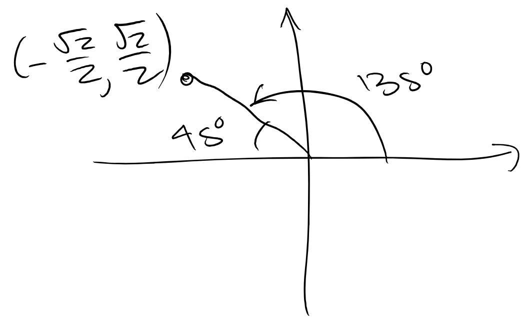


This can be consolidated:

$\frac{\pi}{4} + n\pi$, $0 + n\pi$ $\Rightarrow$ $\left\{ \frac{\pi}{4} + n\pi,\ n\pi \atop n \in \mathbb{Z} \right\}$

5.6 #11) Find $\sin(67.5^\circ)$

$2(67.5^\circ) = 135^\circ$



$$\sin(67.5^\circ) = \sin\left(\frac{135^\circ}{2}\right)$$

$$= \pm\sqrt{\frac{1-\cos(135^\circ)}{2}}$$

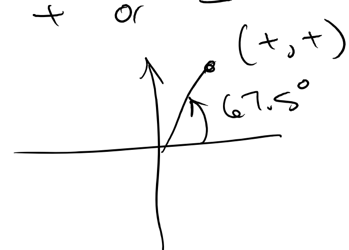
Use $\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{2}}$

$$= \pm\sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}}$$

$$= \pm\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}\cdot\frac{2}{2}}$$

$$= \pm\sqrt{\frac{2+\sqrt{2}}{4}} = \pm\frac{\sqrt{2+\sqrt{2}}}{2}$$

Need to decide + or −



$67.5^\circ \in$ QI, so choose +

$$\sin(67.5^\circ) = \boxed{+\frac{\sqrt{2+\sqrt{2}}}{2}}$$

5.5 #25) Prove. $\sin 4x = 4\sin x\cos x\cos 2x$

5.5 # 49 Express as a trig function of x

Recall:

$\sin(2x) = 2\sin x \cos x$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(2x) = \cos^2 x - \sin^2 x$

$$\sin(4x) = \sin(2\underbrace{(2x)}_{\theta})$$

$$= 2\sin\underbrace{2x}_{\theta}\cos\underbrace{2x}_{\theta}$$

$$= 2(2\sin x\cos x)(\cos^2 x - \sin^2 x)$$