8.3: Conditional Probability, Intersection, and Independence

Conditional probability:

Consider the probability that a house will be flooded during a given year. Would you expect this probability to be different if you only considered houses that were located in a 50-year flood plain? Yes!

Example 1: Draw a single card from a standard 52-card deck.

a. What is the probability that you draw a jack?

J = set of jacks

$n(J) = 4$

$n(S) = 52$

$P(J) = \frac{4}{52} = \boxed{\frac{1}{13}}$

b. New information….Given that you drew a face card (K,Q,J), what is the probability that it is a jack?

Now, the only possibilities are the K, Q, J of each suit

New sample space has $3 \cdot 4 = 12$

prob. of getting a jack is $\frac{4}{12} = \boxed{\frac{1}{3}}$

"given"

Notation: $P(A \mid B)$ denotes the probability of A given that B occurs.

Conditional probability definition:
The probability of A given that B occurs is
$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$. $\quad (P(B) \neq 0)$

Prob. of A given B is the probability of the intersection divided by the probability of the given.

c. Use the conditional probability definition to determine the probability that a jack is drawn, given that the card is a face card.

$n(S) = 52$

F = set of face cards

$n(F) = 3 \cdot 4 = 12$

$P(F) = \frac{12}{52}$

$J \cap F = \{J♡, J♢, J♤, J♧\}$

$n(J \cap F) = 4$

J = set of jacks

Want to find $P(J \mid F)$.

$P(J \mid F) = \dfrac{P(J \cap F)}{P(F)} = \dfrac{4/52}{12/52}$

$= \frac{4}{\cancel{52}} \cdot \frac{\cancel{52}}{12}$

$= \frac{4}{12} = \boxed{\frac{1}{3}}$

Example 2: Draw a single card from a standard 52-card deck. What is the probability of drawing the ace of diamonds given that the card is red?

$A = \{A\diamondsuit\}$

R = set of red cards

$n(R) = 26$

$$P(A|R) = \boxed{\frac{1}{26}}$$

Example 3: When rolling a single die, what is the probability of rolling a prime given that the number rolled is even?

Changing sample space:

Even numbers are $\{2,4,6\}$

which of these are prime? $\{2\}$

$$\boxed{\frac{1}{3}}$$

Using cond. Prob formula

or you can use formula:

$S = \{1,2,3,4,5,6\}$

$E = \{2,4,6\}$

B: Prime numbers

$B = \{2,3,5\}$

$B \cap E = \{2\}$

$$P(B|E) = \frac{P(B \cap E)}{P(E)} = \frac{1/6}{3/6} = \boxed{\frac{1}{3}}$$

Example 4: In a test conducted by the U.S. Army, it was found that of 1000 new recruits, 680 men and 320 women, 57 of the men and 3 of the women were red-green color-blind. Given that a recruit selected at random from this group is red-green color-blind, what is the probability that the recruit is a male?

M = males

C = color-blind people

$n(C) = 57 + 3 = 60$

$$P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{57/1000}{60/1000} = \boxed{\frac{57}{60}} \approx \boxed{0.95}$$

The product rule for intersections of events:

Recall: The definition for conditional probability:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

so $\frac{\text{Prob of the intersection}}{\text{Prob of the given}}$

multiply both sides by P(A):

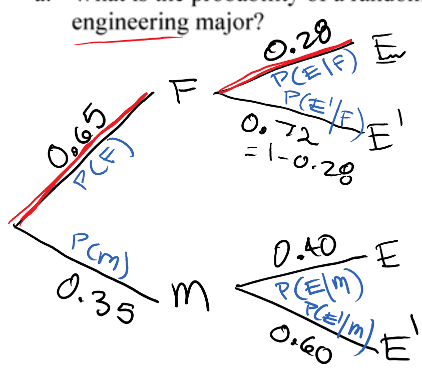
$P(B|A)P(A) = P(B \cap A)$

> Product rule:
> For events A and B with nonzero probabilities
>
> $P(A \cap B) = P(B \mid A)P(A)$

Probability Trees

Example 5: In a certain class, 65% of the students are female. 40% of the males and 28% of the females are engineering majors.

a. What is the probability of a randomly selected student being female and an engineering major?

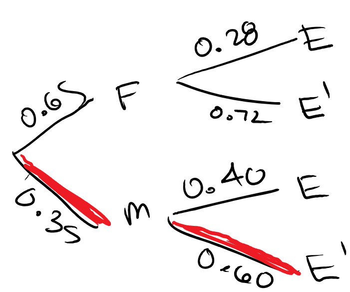


F = females
m = males
E = engineering majors

From product rule,
$P(F \cap E) = P(E|F)P(F)$
Also $P(F \cap E) = P(F|E)P(E)$

multiply along the branch of tree that includes F and E. $P(F \cap E) = 0.65(0.28)$
$= \boxed{0.182}$

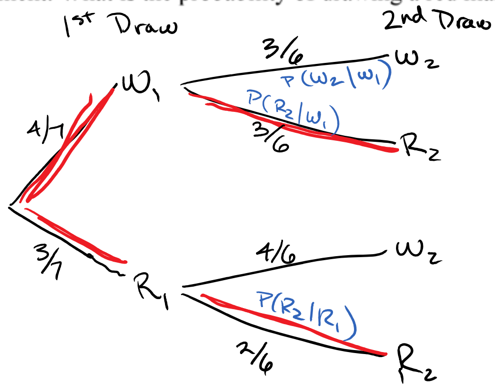
b. What is the probability of a randomly selected student being male and a non-engineering major?



$P(m \cap E') = 0.35(0.60)$
$= \boxed{0.21}$

Example 6: A box contains 4 white marbles and 3 red marbles. Two marbles are drawn in succession without replacement. What is the probability of drawing a red marble on the second draw?

W W W W
R R R



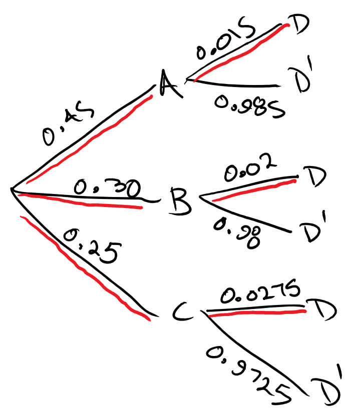
Note:
$P(R_2|R_1) = \frac{2}{6} = \frac{1}{3}$
$P(R_2) = \frac{3}{7}$
so R_1 and R_2 are dependent

$P(R_2) = P(R_2 \cap W_1) + P(R_2 \cap R_1)$

$= \frac{4}{7}\left(\frac{3}{6}\right) + \frac{3}{7}\left(\frac{2}{6}\right) = \frac{12}{42} + \frac{6}{42} = \frac{18}{42} = \boxed{\frac{3}{7}}$

Example 7: A certain type of camera is manufactured in three locations. Plants A, B, and C supply 45%, 30%, and 25%, respectively, of the cameras. The quality-control department of the company has determined that 1.5% of the cameras produced by plant A, 2% of the cameras produced by plant B and 2.75% of the cameras produced by plant C are defective. What is the probability that a randomly selected camera is defective?

D = camera is defective



$P(D) = 0.45(0.015) + 0.30(0.02) + 0.25(0.0275)$

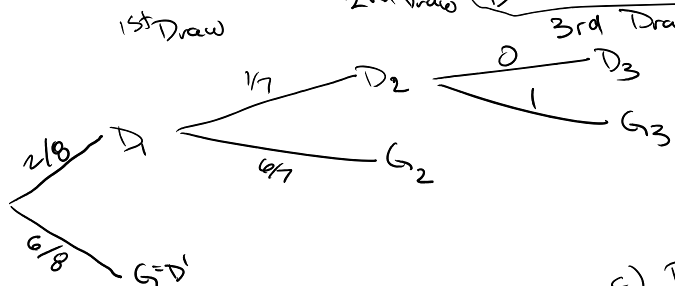
$= \boxed{0.019625}$

Example 8: A box contains eight 9-volt transistor batteries, of which two are known to be defective. The batteries are selected one at a time without replacement and tested until a nondefective one is found. What is the probability that the number of batteries tested is

a. one?
b. two?
c. three?

G = good batteries
D = defective batteries

$\boxed{\cancel{DD}\ GGGGGG}$



a) Prob of testing 1 battery is $6/8 = \frac{3}{4}$.

b) Prob. of testing 2 batteries is
$P(D_1 \cap G_2) = \frac{2}{8} \cdot \frac{6}{7} = \frac{12}{56} = \boxed{\frac{3}{14}}$

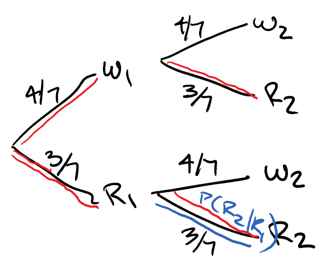
c) Prob of testing 3 batteries is
$P(D_1 \cap D_2 \cap G_3) = \frac{2}{8}\left(\frac{1}{7}\right)(1) = \frac{2}{56} = \boxed{\frac{1}{28}}$

Independence of events:

Example 9: (Compare with Example 6) A box contains 4 white marbles and 3 red marbles. Two marbles are drawn in succession, this time replacing the first before drawing the second.

a. What is the probability of drawing a red marble on the second draw?
b. What is the probability of drawing a red marble on the second draw given that a red marble was draw on the first draw?

$\boxed{\begin{matrix} WWWW \\ RRR \end{matrix}}$



a) $P(R_2) = \frac{4}{7} \cdot \frac{3}{7} + \frac{3}{7} \cdot \frac{3}{7}$
$= \frac{12}{49} + \frac{9}{49} = \frac{21}{49} = \boxed{\frac{3}{7}}$

b) $P(R_2|R_1) = \boxed{\frac{3}{7}}$

Notice:
$P(R_2) = P(R_2|R_1)$.
The fact that R_1 is "given" does not affect the probability that R_2 happens.

Two events are said to be *independent* of the outcome of one does not affect the outcome of the other. If they are not independent, then they are said to be *dependent*.

> Independent Events:
> Events A and B are independent events if and only if:
>
> - $P(A \mid B) = P(A)$ or, equivalently,
> - $P(B \mid A) = P(B)$ or, equivalently,
> - $P(A \cap B) = P(A)P(B)$.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$\underbrace{P(A \mid B)}_{\text{equal to } P(A) \text{ if ind.}} P(B) = P(A \cap B).$$

If A and B are independent, then $P(A \mid B) = P(A)$. Replace $P(A \mid B)$ by $P(A)$. So we get $P(A)P(B) = P(A \cap B)$

Example 10: Draw a single card from a standard deck. Show whether the following pairs of events are independent or dependent.

a. Drawing a heart and drawing a face card.
b. Drawing a king and drawing a queen.

a) Are the events "draw a heart" and "draw a face card independent"?

F: face card
H: heart

$$P(F) = \frac{12}{52} = \frac{6}{26} = \frac{3}{13}$$

$$P(F \mid H') = \frac{9}{39} = \frac{3}{13}$$

(39 non-hearts in deck)
9 of the 39 non-hearts are face cards

$\boxed{P(F) = P(F \mid H'), \text{ so } F \text{ and } H \text{ are independent}}$

b) Are "drawing a King" and "drawing a queen" independent events

K: King
Q = queen

$$P(K) = \frac{4}{52} = \frac{1}{13}$$

Suppose we know for sure the card drawn is not a queen.
What is the prob. she got a king?

$$P(K \mid Q') = \frac{4}{48} = \frac{1}{12}$$

(there are 48 non-queens in the deck
4 of the non-queens are kings)

$\boxed{P(K) \neq P(K \mid Q'), \text{ so } K \text{ and } Q \text{ are dependent}}$

Another way to test K and Q for independence:

$P(K \cap Q) = 0$
(because $K \cap Q = \emptyset$)

$P(K) = \frac{4}{52} = \frac{1}{13}$ $P(Q) = \frac{4}{52} = \frac{1}{13}$

$P(K)P(Q) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169} \neq 0$

$P(K \cap Q) \neq P(K)P(Q)$, so K and Q are dependent

Example 11: A survey conducted found that of 2000 women, 680 were heavy smokers and 50 had emphysema. Of those who had emphysema, 42 were also heavy smokers. Using this data, determine whether the events "being a heavy smoker" and "having emphysema" were independent events.

E: having emphysema

H: being a heavy smoker

Test for independence: Is $P(H \cap E) = P(H)P(E)$?

$$P(H \cap E) = \frac{42}{2000} = 0.021$$

$$\left.\begin{array}{l} P(H) = \frac{680}{2000} = 0.34 \\ P(E) = \frac{50}{2000} = 0.025 \end{array}\right\} P(H)P(E) = 0.34(0.025) = 0.0085$$

compare to $P(H \cap E) = 0.021$

They're not equal!

$P(H \cap E) \neq P(E)P(H)$,

so E and H are not independent (they are dependent).

Independence of more than two events:

If $E_1, E_2, \ldots, E_n$ are independent, then

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1)P(E_2)\ldots P(E_n).$$

Example 12: A certain loudspeaker system has four components: a woofer, a midrange, a tweeter, and an electrical crossover. It has been determined that on the average 1% of the woofers, 0.8% of the midranges, 0.5% of the tweeters, and 1.5% of the crossovers are defective. Determine the probability that a randomly chosen loudspeaker is not defective. Assume that the defects in the different types of components are unrelated.