7.4: Permutations and Combinations

Permutations:

Example 1: Six horses are entered in a race. Assuming no ties, in how many possible ways can they finish first, second, and third?

$$\underset{1^{st}}{\underline{6}} \cdot \underset{2^{nd}}{\underline{5}} \cdot \underset{3^{rd}}{\underline{4}} = \boxed{120}$$

This is a "permutation", or rearrangement.

Example 2: An Olympic event has ten competitors. In how many ways can the gold, silver, and bronze medals be awarded (assuming no ties)?

$$\underset{\text{Gold}}{\underline{10}} \cdot \underset{\text{silver}}{\underline{9}} \cdot \underset{\text{Bronze}}{\underline{8}} = \boxed{720}$$

> Permutations of *n* Objects Taken *r* at a Time:
>
> The number of permutations of *n* objects taken *r* at a time without repetition is given by
>
> $$P_{n,r} = \frac{n!}{(n-r)!}$$
>
> Note: $P_{n,n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$. So there are $n!$ ways to arrange *n* objects.

So for the last example, we have $P_{10,3} = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10\cdot9\cdot8\cdot\cancel{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}}{\cancel{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}}$

$n = 10, \; r = 3$

$= 10\cdot9\cdot8 = \boxed{720}$

Example: $P_{5,2} =$ $P_{5,2} = \frac{5!}{3!} = \frac{5\cdot4\cdot\cancel{3\cdot2\cdot1}}{\cancel{3\cdot2\cdot1}} = \boxed{20}$

$P_{8,5} = \frac{8!}{3!} = 6720$

$P_{7,7} = \boxed{5040}$ Note: $P_{7,7} = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1} = 7!$ $= \boxed{5040}$

$P_{4,3} = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4\cdot3\cdot2 = \boxed{24}$

Other notations: $P_{n,r}$ $\quad {}_nP_r$ $\quad P(n,r)$ $\quad P_r^n$

Combinations:

Example 3: A student group with five officers must form a three-member committee. How many different committees can be formed?

$$\frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1} = \frac{60}{6} = \boxed{10}$$

This is what we call a *combination* problem rather than a *permutation* problem.

Notice that in this situation, the order does not matter. In other words, "Joe, Mary, Sue" is the same committee as "Mary, Sue, Joe".

<u>Combinations of *n* Objects Taken *r* at a Time</u>:

The number of combinations of *n* objects taken *r* at a time without repetition is given by

$$C_{n,r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Note: $C_{n,n} = \dfrac{n!}{n!(n-n)!} = \dfrac{n!}{n!0!} = \dfrac{n!}{n!(1)} = 1.$

Back to our committees…

$n=5$
$r=3$

$$C_{5,3} = \frac{5!}{3!2!} = \frac{5\cdot 4\cdot \cancel{3\cdot 2\cdot 1}}{\cancel{3\cdot 2\cdot 1}\cdot 2\cdot 1} = \frac{20}{2} = \boxed{10}$$

$$= \frac{5!}{3!(5-3)!}$$

Note: $C_{5,2} = \boxed{10}$ same

Example 4: $C_{6,2} =$

$$C_{6,2} = \frac{6!}{2!4!} = \frac{6\cdot 5\, \cancel{4!}}{2\cdot 1\, \cancel{4!}} = \boxed{15}$$

$C_{7,4} = \boxed{35}$ Note: $C_{7,3} = \boxed{35}$ also

$C_{10,9} = \boxed{10}$ Note: $C_{10,1} = \boxed{10}$

Other notations: ${}_nC_r$ $\quad C_{n,r}$ $\quad C(n,r)$ $\quad C_r^n$ $\quad \binom{n}{r}$

Example 5: A student group with five members must choose a president, vice-president, and treasurer. In how many ways can this be done?

$$\underset{\text{Pres}}{\underline{5}} \cdot \underset{\text{VP}}{\underline{4}} \cdot \underset{\text{Treas}}{\underline{3}} = \boxed{60}$$

or $P_{5,3} = \boxed{60}$

Example 6: An art museum has a collection of 7 sculptures by a particular artist. There is only room to display four of the sculptures at a time. In how many different ways can four sculptures be chosen to display?

$C_{7,4} = \boxed{35}$

> **Very Important:**
> - If order matters, use permutations.
> - If order does not matter, use combinations.

Example 7: Consider a standard 52-card deck.

a. How many 5-card hands contain 5 hearts?

$C_{13,5} = \boxed{1287}$

a.5) How many 5-card hands are possible?

$C_{52,5} = \boxed{2\,598\,960}$

b. How many 5-card hands will contain exactly 2 aces and 2 queens?

c. How many 5-card hands will contain 2 hearts and 3 clubs?