3.2: Compound Interest

If at the end of a payment period, the interest due is reinvested at the same rate, then the interest as well as the principal will earn interest. This is called *compound interest*. The interest is paid into the account at the end of each compounding period.

Example 1: Suppose you invest $1000 compounded quarterly at an annual interest rate of 8%. How much money will you have after one year?

quarterly = 4 times per year

end of 1st Quarter: $P = \$1000$, $r = 0.08$, $t = \frac{1}{4}$

$A = P(1+rt)$

$= \$1000(1+0.08(\frac{1}{4}))$

$= \$1000(1.02) = \$1020$

end of 2nd Quarter: $P = \$1020$, $r = 0.08$, $t = \frac{1}{4}$

$A = P(1+rt)$

$A = \$1020(1+0.08(\frac{1}{4})) = \$1020(1.02)$

$= \$1040.40$

3rd Qtr: $P = \$1040.40$, $r = 0.08$, $t = \frac{1}{4}$

$A = \$1040.40(1+0.08(\frac{1}{4}))$

$= \$1040.40(1.02)$

$\approx \$1061.21$

4th Qtr: $P = \$1061.21$

$A = \$1061.21(1.02)$

$\approx \$1082.43$

Note:

Q1: $A = 1000(1.02)$

Q2: $A = 1000(1.02)(1.02)$

$= 1000(1.02)^2$

Q3: $A = 1000(1.02)^3$

Q4: $A = 1000(1.02)^4$

Compound Interest:

$$A = P(1+i)^n$$

$$= P\left(1+\frac{r}{m}\right)^n$$

$$= P\left(1+\frac{r}{m}\right)^{mt}$$

where

$i = \frac{r}{m}$ is the interest rate per compounding period

$r =$ annual interest rate

$m =$ number of compounding periods per year

$n =$ total number of compounding periods

$P =$ principal (present value)

$A =$ amount (future value) at the end of n compounding periods.

Daily Compounding: $m = 365$

Monthly Compounding: $m = 12$

semi-annual comp.: $m = 2$

quarterly: $m = 4$

annually: $m = 1$

Example 2: What is the future value of $1000 after 8 years at 6% compounded monthly?

$A = P(1+i)^n$

$A = P\left(1+\frac{r}{m}\right)^n$

$i = \frac{r}{m} = \frac{0.06}{12}$

$r = 0.06$

$m = 12$

$A = ?$

$P = \$1000$

$n = mt$

$= 12(8) = 96$

$A = \$1000\left(1+\frac{0.06}{12}\right)^{96}$

$A = \$1614.14$

Example 3: How much should I invest now at 4% interest compounded monthly in order to have $10,000 in 6 years?

Example 4: You decide to invest some money so that you will have $1,000,000 on your 75th birthday. At 8% compounded quarterly, how much should you invest on your 25th birthday?

Example 5: How long will it take \$5,000 to grow to \$7,000 if it is invested at 8% compounded monthly?

Example 6: How long will it take money to double if it is invested at 7.5% compounded monthly?

Continuous compound interest:

In calculus, a fundamental topic is the *limit*, or limiting value of a function. If we allow the number of compounding periods per year to increase toward infinity, the amount A approaches the limiting value $A = Pe^{rt}$. The number e is a constant, $e \approx 2.71828$. The number e is irrational—it cannot be written as a fraction of integers, or as a decimal that ends or repeats.

e can be defined as the limiting value of $\left(1+\frac{1}{x}\right)^x$ as x approaches ∞.

Start with the compound interest formula:

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

Substitute $x = \frac{m}{r}$ and then rearrange/simplify:

$$A = P\left[\left(1+\frac{1}{x}\right)^x\right]^{rt}$$

As $x \to \infty$, $\left(1+\frac{1}{x}\right)^x \to e$. This gives us the formula for continuous compound interest.

<u>Continuous Compound Interest</u>:

If principal P is compounded continuously at the annual interest rate r, then the amount at the end of t years is

$$A = Pe^{rt}.$$

<u>Example 7:</u> How much must be invested now to have \$60,000 available in 10 years, if it is invested at 7% compounded (a) monthly? (b) continuously?

<u>Example 8:</u> How long will it take \$5,000 to grow to \$7,000 if it is invested at 8% compounded continuously?

Effective rates:

The effective rate, sometimes called the *annual percentage yield*, converts a compound interest rate to an equivalent simple interest rate. This allows us to compare interest rates which have different compounding periods.

Annual Percentage Yield (APY):

The annual percentage yield (APY), or effective rate, is given by

$$APY = r_e = \left(1 + \frac{r}{m}\right)^m - 1,$$

where

$r =$ annual interest rate

$m =$ number of compounding periods per year.

For interest compounded continuously, the APY is

$$APY = r_e = e^r - 1.$$

<u>Example 9:</u> What is the annual percentage yield for money invested at 6% compounded quarterly?

<u>Example 10:</u> Which investment is better, Note A at 9% compounded monthly or Note B at 9.2% compounded semiannually?