3.3: Future Value of an Annuity; Sinking Funds

An *annuity* is any sequence of equal, equally spaced, periodic payments. An *ordinary annuity* is an annuity where the payments are made at the end of each time interval. Ordinary annuities are the only type of annuity we will study. The *future value* of an annuity is the sum of all payments plus interest.

Example 1: Over three years, you deposit $100 into an account every 6 months. This account pays 6% interest compounded semiannually. How much money will be in the account at the end of three years?

Future Value of an Ordinary Annuity:

$$FV = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

where

$PMT =$ periodic payment (made at end of period)

$i = \frac{r}{m} =$ rate per period

$n =$ number of payments (periods)

$FV =$ future value (amount)

Example 2: What is the value of an annuity at the end of 10 years if $1,000 is deposited every 3 months into an account earning 8% interest compounded quarterly? How much of this is interest?

$$FV = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

$FV = ?$

$PMT = 1000$

$i = \frac{r}{m} = \frac{0.08}{4}$

$r = 0.08$

$m = 4$

$n = mt = 4(10) = 40$

$$= 1000\left[\frac{\left(1+\frac{0.08}{4}\right)^{40} - 1}{\frac{0.08}{4}}\right]$$

$$FV = \$60\,401.98$$

Any account that is established for the purpose of accumulating funds to meet future obligations is called a *sinking fund*.

Example 3: The parents of a newborn baby set up an account to cover the cost of college. On the child's birthday each year, starting on the 1st and ending on the 18th, they deposit money into the account, which pays 7% compounded annually. How much should they deposit annually in order to have $100,000 available for college on the child's 18th birthday?

Warning: To use this formula, the periodic payments need to be made at the end of each compounding period!

For example, you can't have monthly payments and quarterly compounding periods.

Example 4: At the age of 25, Joe decides to start saving for retirement by depositing \$300 each month into an account earning 6% interest compounded monthly. He does this until he turns 40. Then he quits making payments and lets his money sit, still earning 6% compounded monthly, until he turns 65.

a. How much will Joe have when he turns 65?

b. Tom gets a late start and doesn't start saving until he turns 40. How much will he have to deposit monthly (same interest rate) in order to have the same amount as Joe at the age of 65?

c. How much interest is earned during Joe's first year of saving? During his last year of saving? During his 15th year of saving?

(a) Age 25 – 40:

$$FV = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

$FV = ?$

$i = \frac{r}{m} = \frac{0.06}{12}$

$n = mt = 12(15) = 180$

$40 - 25 = 15$

$PMT = \$300$

$$FV = \$300\left[\frac{\left(1+\frac{0.06}{12}\right)^{180} - 1}{\frac{0.06}{12}}\right]$$

$FV = \$87\,245.61$ amount Joe has at age 40

Age 40 to 65:

Use compound interest formula:

$$A = P\left(1+\frac{r}{m}\right)^n$$

$A = ?$

$P = \$87\,245.61$

$i = \frac{r}{m} = \frac{0.06}{12}$

$n = mt = 12(25) = 300$

$t = 65 - 40 = 25$ years

$$A = \$87\,245.61\left(1+\frac{0.06}{12}\right)^{300}$$

$$A = \boxed{\$389\,549.01}$$

(amount Joe has at age 65)

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(b) Tom: Age 40 to 65:

$$FV = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

$FV = \$389\,549.01$

$PMT = ?$

$i = \frac{r}{m} = \frac{0.06}{12}$

$n = mt \quad 12(25) = 300$

$$\$389\,549.01 = PMT\left[\frac{\left(1+\frac{0.06}{12}\right)^{300} - 1}{\frac{0.06}{12}}\right]$$

$$PMT = \boxed{\$562.12}$$ (amt Tom needs to pay monthly to catch up w/ Joe)

Note:

Total Amt paid by Joe: $\$300(180) = \$54000$

Total Amt paid by Tom: $\$562.12(300) = \$168\,636$