4.4: Matrices-Basic Operations

We will learn how to add matrices and how to multiply a matrix by a number (scalar).

Equality:

Two matrices are *equal* if they are the same size *and* all the corresponding elements are equal.

Example 1:

$$A=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix},\ B=\begin{bmatrix}1&2\\3&4\end{bmatrix},\ C=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix},$$

$B = D$

$$D=\begin{bmatrix}1&2\\3&4\end{bmatrix},\ E=\begin{bmatrix}1&1\\1&1\end{bmatrix},\ F=\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}$$

$E \neq F$ (different sizes)

Addition:

Matrices can be added only if they are the same size. In this case, we add the corresponding elements in the matrices.

Example 2:

$$\underset{2\times2}{\begin{bmatrix}1&3\\5&7\end{bmatrix}}+\underset{2\times1}{\begin{bmatrix}1\\7\end{bmatrix}} \neq \boxed{\text{cannot be added}}$$

(not defined)

$$\begin{bmatrix}2&4\\4&2\end{bmatrix}+\begin{bmatrix}1&2\\3&4\end{bmatrix}=\boxed{\begin{bmatrix}3&6\\7&6\end{bmatrix}}$$

$$\underset{3\times1}{\begin{bmatrix}-1\\2\\3\end{bmatrix}}+\underset{3\times1}{\begin{bmatrix}4\\-5\\6\end{bmatrix}}=\begin{bmatrix}-1+4\\2-5\\3+6\end{bmatrix}=\boxed{\begin{bmatrix}3\\-3\\9\end{bmatrix}}$$

Addition of numbers is associative and commutative. Addition of matrices is associative and commutative also.

- <u>Commutative</u>: $A+B=B+A$
- <u>Associative</u>: $(A+B)+C=A+(B+C)$

A *zero matrix* is a matrix with zero in all positions. The following are zero matrices of different sizes:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2x2 3x2

(opposite)

The *negative of a matrix A*, denoted $-A$, is the matrix with all elements that are the opposites of the corresponding elements in the matrix *A*.

Example 3:

$$A = \begin{bmatrix} 5 & -12 \\ -7 & 6 \end{bmatrix}$$

$$-A = \begin{bmatrix} -5 & 12 \\ 7 & -6 \end{bmatrix}$$

Subtraction:

As with addition, subtraction can be performed only if matrices are the same size. The difference $A-B$ is defined to be $A+(-B)$. So to subtract, we just subtract the corresponding elements.

Example 4:

$$\begin{bmatrix} 1 & 2 & -6 \\ -3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 0 & -2 & -2 \\ 4 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 1-0 & 2-(-2) & -6-(-2) \\ -3-4 & 4-6 & 5-5 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & 4 & -4 \\ -7 & -2 & 0 \end{bmatrix}}$$

Multiplication of a matrix by a number:

The product of a number *k* and a matrix *M*, denoted by *kM*, is the matrix formed by multiplying each element of *M* by *k*. This is often called *scalar multiplication*.

Example 5:

$$4\begin{bmatrix} 1 & 3 \\ -2 & 7 \\ 0 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 12 \\ -8 & 28 \\ 0 & -16 \end{bmatrix}}$$

Product of a row matrix and a column matrix (in that order):

The product of a $1\times n$ row matrix A and an $n\times 1$ column matrix is the 1×1 matrix given by

$$AB = \underbrace{\begin{bmatrix} a_1 & a_2 \cdots a_n \end{bmatrix}}_{1\times n} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} a_1b_1 + a_2b_2 + \cdots + a_nb_n \end{bmatrix}}_{1\times 1}.$$

<u>Note</u>: For this formula to hold, they must be in this order: row$\times$column. If they are in the other order (column$\times$row), you get a different result. We'll see one like this later.

<u>Note</u>: The number of elements in the row and column must be the same in order for the multiplication to be defined.

<u>**Example 6:**</u>

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 & -5 \end{bmatrix}}_{1\times 4} \underbrace{\begin{bmatrix} -4 \\ 0 \\ 2 \\ -4 \end{bmatrix}}_{4\times 1} = \begin{bmatrix} 1(-4) - 2(0) + 3(2) - 5(-4) \end{bmatrix} = \begin{bmatrix} -4 - 0 + 6 + 20 \end{bmatrix} = \underbrace{\boxed{\begin{bmatrix} 22 \end{bmatrix}}}_{1\times 1}$$

$$\begin{bmatrix} 21 & -32 & 19 \end{bmatrix} \begin{bmatrix} 11 \\ 23 \\ 13 \end{bmatrix} = \begin{bmatrix} 21(11) - 32(23) + 19(13) \end{bmatrix} = \boxed{\begin{bmatrix} -258 \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} 3 & -2 & 5 & 3 \end{bmatrix}}_{1\times 4} \underbrace{\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}}_{3\times 1} = \boxed{\text{not possible}}$$

Matrix multiplication:

If A is an $m\times p$ and B is a $p\times n$ matrix, then the product of these is denoted AB and it is an $m\times n$ matrix.

must match

$$(m\times \cancel{p})(\cancel{p}\times n) = (m\times n)$$

The entries in the matrix AB are formed as follows: the element in the ith row and jth column is the product of the ith row of A with the jth column of B.

<u>*Important Note*</u>: If the number of columns of A is not equal to the number of rows of B, the product AB is not defined! The matrices cannot be multiplied!!

Example 7:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \underset{R_1C_1}{9} & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$$

$2 \times \cancel{3}$ $\quad \cancel{3} \times 2$ $\quad 2 \times 2$

$=$

$R_1 \cdot C_1$: $2(1) + 1(4) + 3(1)$
$= 9$

$$AB = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & \underset{R_1C_2}{1} \\ \_\_ & \_\_ \end{bmatrix}$$

R1 C2: $2(0) + 1(-2) + 3(1)$
$= 0 - 2 + 3$
$= 1$

$$AB = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ \underset{R_2C_1}{-9} & \_\_ \end{bmatrix}$$

R2C1: $0(1) - 2(4) - 1(1)$
$= 0 - 8 - 1$
$= -9$

$$AB = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & 1 \\ -9 & 3 \end{bmatrix}}$$

R2C2: $0(0) - 2(-2) - 1(1)$
$0 + 4 - 1 = 3$

Example 8:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 10 \\ 5 & 8 \end{bmatrix}}$$

$2 \times \cancel{2}$ $\quad \cancel{2} \times 2$ $\quad 2 \times 2$

match

R1C1 $1(3) + 2(-1) = 3 - 2 = 1$

R1C2: $1(2) + 2(4) = 2 + 8 = 10$

R1 C1: $2(3) + 1(-1) = 6 - 1 = 5$

R2C2: $2(2) + 1(4) = 4 + 4$
$= 8$

Example 9:

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}4 & 5 & 6\end{bmatrix} = \begin{bmatrix}4 & 5 & 6\\8 & 10 & 12\\12 & 15 & 18\end{bmatrix}$$

$3 \times \cancel{1}$ $\cancel{1} \times 3$ match 3×3

R1 C1: 1(4) = 4
R1 C2: 1(5) = 5
R1 C3 1(6) = 6
R2 C1: 2(4) = 8

Example 10:

$$\begin{bmatrix}-1 & 3\\4 & 2\\-2 & 5\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\6 & 1 & -2\end{bmatrix} = \begin{bmatrix}17 & 1 & -9\\16 & 10 & 8\\28 & 1 & -16\end{bmatrix}$$

$3 \times \cancel{2}$ $\cancel{2} \times 3$ match 3×3

R1 C1: $-1(1) + 3(6)$
$= -1 + 18 = 17$

Example 11:

$$\begin{bmatrix}1 & 3 & 1\\2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} = \boxed{\text{not possible}}$$

2×3 2×3 $\neq$

Example 12:

$$\underset{A}{\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}}\underset{B}{\begin{bmatrix}2 & 6\\-1 & -3\end{bmatrix}} = \boxed{\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}}$$

R1 C1: $1(2) + 2(-1) = 0$
R1 C2 $1(6) + 2(-3) = 6 - 6 = 0$

same matrices, different order

Ex 13:

$$\underset{B}{\begin{bmatrix}2 & 6\\-1 & -3\end{bmatrix}}\underset{A}{\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}} = \boxed{\begin{bmatrix}20 & 40\\-10 & -20\end{bmatrix}}$$

Note: so $AB \neq BA$

Example 13:

$$\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} =$$

(worked out on previous page)

Example 14:

$$\begin{bmatrix} 2 & -4 & 3 \\ -3 & 1 & 5 \\ 10 & -2 & 7 \end{bmatrix}^2 = \begin{bmatrix} 2 & -4 & 3 \\ -3 & 1 & 5 \\ 10 & -2 & 7 \end{bmatrix}\begin{bmatrix} 2 & -4 & 3 \\ -3 & 1 & 5 \\ 10 & -2 & 7 \end{bmatrix}$$

R1C1: $2(2) - 4(-3) + 3(10)$
$= 46$

$$= \boxed{\begin{bmatrix} 46 & -18 & 7 \\ 41 & 3 & 31 \\ 96 & -56 & 69 \end{bmatrix}}$$