4.5: Inverse of a Square Matrix

Identity matrix for multiplication:

For real numbers, 1 is the identity for multiplication.

$$1 \cdot a = a \cdot 1 = a$$

Is there an identity for matrix multiplication? i.e. is there a matrix I such that $MI = IM = M$?

In general, no such thing. (Can't reverse multiplication order for non-square matrices)

However, for square matrices, there is such an identity.

For $n \times n$ matrices, I is the matrix with 1 on the principal diagonal and zeros elsewhere.

Examples of Identity Matrices:

$$I_2 = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{2\times 2} \qquad I_3 = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3\times 3} \qquad I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Ex.:

$$\begin{bmatrix} 1 & 4 & -7 \\ 2 & 5 & 8 \\ 3 & 6 & -9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Similarly

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & -7 \\ 2 & 5 & 8 \\ 3 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -7 \\ 2 & 5 & 8 \\ 3 & 6 & -9 \end{bmatrix}$$

Inverses:

Every real number except 0 has a multiplicative inverse.

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$$

← multiplicative identity

(multiplicative)

The Inverse of a Square Matrix:

Let M be an $n \times n$ square matrix and I be the $n \times n$ identity matrix. If there exists a matrix M^{-1} such that $M^{-1}M = MM^{-1} = I$, then M^{-1} is the inverse of M.

Example 1: Verify that $\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$ are inverses of one another.

$$\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

R1C1: $5\left(\frac{1}{2}\right)+3\left(-\frac{1}{2}\right) = \frac{5}{2}-\frac{3}{2} = \frac{2}{2} = 1$

R1C2: $5\left(-\frac{3}{2}\right)+3\left(\frac{5}{2}\right) = -\frac{15}{2}+\frac{15}{2} = 0$

R1C1: $1\left(\frac{1}{2}\right)+1\left(-\frac{1}{2}\right) = \frac{1}{2}-\frac{1}{2} = 0$

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore they are inverses.

How to find the inverse:

- To find the inverse of a matrix *M*, start by creating an augmented matrix $\left[M|I\right]$ by placing the appropriate-sized identity matrix to the right of the vertical line.
- Then row-reduce the augmented matrix until the identity matrix appears to the left of the vertical line. Then M^{-1} is to the right of the vertical line. In other words, row-reduce your augmented matrix until it looks like $\left[I|M^{-1}\right]$.
- If a zero row appears to the left of the vertical line, then M^{-1} does not exist.

Example 2: Find the inverse of $M = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, if it exists.

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \underset{\sim}{R_1 \longleftrightarrow R_2} \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ \boxed{2} & 3 & 1 & 0 \end{array}\right] \underset{\sim}{-2R_1+R_2 \rightarrow R_2}$$

want 1

want 0

want 0

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & \boxed{-1} & 1 & -2 \end{array}\right] \underset{\sim}{-1R_2 \rightarrow R_2} \left[\begin{array}{cccc} 1 & \boxed{2} & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

want 1

$$\underset{\sim}{-2R_2+R_1 \rightarrow R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\text{So } \boxed{M^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}}$$

To check, make sure $MM^{-1} = I$

Example 3: Find the inverse of $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, if it exists.

Example 4: Find the inverse of $M = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$, if it exists.

Goal: $\underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \end{array}\right]}_{M^{-1}}$

want 1

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\xrightarrow{-2R_1+R_3 \to R_3}} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & \enclose{circle}{2} & -1 & 0 & 1 & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right]$$

want 0

$$\underset{\sim}{\tfrac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & \enclose{circle}{-1} & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \enclose{circle}{5} & -2 & -2 & 0 & 1 \end{array}\right] \begin{array}{c} 1R_2+R_1 \to R_1 \\ \sim \\ -5R_2+R_3 \to R_3 \end{array}$$

want 0s

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \enclose{circle}{\frac{1}{2}} & -2 & -\frac{5}{2} & 1 \end{array}\right] \underset{2R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & \enclose{circle}{\frac{1}{2}} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & \enclose{circle}{-\frac{1}{2}} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right]$$

want 1

$$\begin{array}{c} -\tfrac{1}{2}R_3+R_1 \longrightarrow R_1 \\ \sim \\ \tfrac{1}{2}R_3+R_2 \longrightarrow R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right]$$

$$\boxed{M^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}}$$

using determinant

$|M| = D = 2(-1) - 2(-1)$

$= -2 + 2 = 0$

M^{-1} does not exist

want 1

Example 5: Find the inverse of $M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$, if it exists.

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right] \underset{\sim}{\xrightarrow{\frac{1}{2}R_1 \to R_1}} \left[\begin{array}{cc|cc} 1 & 1 & 1/2 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right] \underset{\sim}{\xrightarrow{1R_1 + R_2 \to R_2}}$$

want 0

$$\left[\begin{array}{cc|cc} 1 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 \end{array}\right]$$

want 1

It is impossible to get a 1 in this location!

$\boxed{M^{-1} \text{ does not exist}}$

(M does not have an inverse)

Shortcut for 2×2 matrices:

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the *determinant* of M is $D = ad - bc$. If $D \neq 0$, then M^{-1} exists and is given by

determinant of M is $|M| = ad - bc$

$$M^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ —

Example 6: Find the inverse of $M = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}$, if it exists.

$|M| = D = -5(4) - (-3)(6)$

$= -20 + 18 = -2$

$M^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} \frac{4}{-2} & \frac{3}{-2} \\ \frac{-6}{-2} & \frac{-5}{-2} \end{bmatrix}$

$$\boxed{M^{-1} = \begin{bmatrix} -2 & -\frac{3}{2} \\ 3 & \frac{5}{2} \end{bmatrix}}$$

Example 7: Find the inverse of $M = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$, if it exists.

$$|M| = D = 2(6) - 4(3)$$
$$= 12 - 12$$
$$= 0$$

Determinant is 0, so $\boxed{M^{-1} \text{ does not exist.}}$

Example 8: Find the inverse of $M = \begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix}$, if it exists.

$$|M| = D = 3(8) - 5(2) = 24 - 10 = 14$$

$$M^{-1} = \frac{1}{14}\begin{bmatrix} 8 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 8/14 & -5/14 \\ -2/14 & 3/14 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4/7 & -5/14 \\ -1/7 & 3/14 \end{bmatrix}}$$