## 4.6: Matrix Equations and Systems of Linear Equations

Using the definitions and properties of matrices that we've learned so far will let us use matrices to solve other types of problems. In particular, we can use them to solve systems of linear equations.

Consider an ordinary linear equation, $ax = b$. To solve this, you would multiply both sides by the multiplicative inverse of $a$.

$$ax = b$$
$$a^{-1}ax = a^{-1}b$$
$$1x = a^{-1}b$$
$$x = a^{-1}b$$

We'll solve matrix equations the same way.

> If $A$ is an $n \times n$ square matrix and has an inverse, and $X$ and $B$ are $n \times 1$ column matrices, then the solution to $AX = B$ is given by $X = A^{-1}B$.

Here's why:

$$\underset{(n\times n)(n\times 1)}{AX} = \underset{n\times 1}{B}$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

Important Note: As we have seen previously, matrix multiplication is not commutative (products aren't necessarily equal when we change the order).
So, when $AX = B$, then $X = A^{-1}B$, not $X = BA^{-1}$!!!

**Example 1:** Solve the matrix equation $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$
$$|A| = D = 1(4) - 3(1) = 1$$
$$A^{-1} = \frac{1}{1}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}}_{I}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underset{2\times 2}{\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}}\underset{2\times 1}{\begin{bmatrix} 9 \\ 6 \end{bmatrix}}$$

Scratch

$4(9) - 3(6) = 18$
$-1(9) + 1(6) = -3$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 18 \\ -3 \end{bmatrix}$$

$$\boxed{x_1 = 18,\ x_2 = -3}$$

A system of linear equations can be written as a matrix equation. Then we can use inverses to solve the matrix equation, and this will also give us the solution to the original system.

**Example 2:** Write the system $\begin{cases} x_1 + 2x_2 = 1 \\ x_1 + 3x_2 = 3 \end{cases}$ as a matrix equation.

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

**Example 3:** Write the system $\begin{cases} 2x_1 - x_2 = 6 \\ -2x_1 + 3x_2 - x_3 = -4 \\ 4x_1 + 3x_3 = 7 \end{cases}$ as a matrix equation.

$$\begin{aligned} 2x_1 - x_2 \quad\quad &= 6 \\ -2x_1 + 3x_2 - x_3 &= -4 \\ 4x_1 \quad\quad + 3x_3 &= 7 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix}$$

**Example 4:** Write the matrix equation $\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$ as a system of equations.

$$x_1 + x_2 = 10$$
$$3x_1 - 2x_2 = 20$$

**Example 5:** Solve the system using the inverse of the coefficient matrix.

$$\begin{cases} 7x_1 + 4x_2 = 11 \\ 2x_1 + 3x_2 = -9 \end{cases}.$$

**Example 6:** Solve the system using the inverse of the coefficient matrix.

$$\begin{cases} 3x+2y-z=2 \\ 2x-3y+z=-2 \\ x-y-z=4 \end{cases}$$

Write the matrix equation:

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

Find the inverse:

$$\left[\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Row operations ... (To see the row operations, watch my Matrix Equations video on the math website.)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4/17 & 3/17 & -1/17 \\ 0 & 1 & 0 & 3/17 & -2/17 & -5/17 \\ 0 & 0 & 1 & 1/17 & 5/17 & -13/17 \end{array}\right]$$

So
$$A^{-1} = \begin{bmatrix} 4/17 & 3/17 & -1/17 \\ 3/17 & -2/17 & -5/17 \\ 1/17 & 5/17 & -13/17 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4/17 & 3/17 & -1/17 \\ 3/17 & -2/17 & -5/17 \\ 1/17 & 5/17 & -13/17 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{17} - \frac{6}{17} - \frac{4}{17} \\ \frac{6}{17} + \frac{4}{17} - \frac{20}{17} \\ \frac{2}{17} - \frac{10}{17} - \frac{52}{17} \end{bmatrix} = \begin{bmatrix} -2/17 \\ -10/17 \\ -60/17 \end{bmatrix}$$

$$\boxed{(x, y, z) = \left(-\frac{2}{17}, -\frac{10}{17}, -\frac{60}{17}\right)}$$