6.2: The Simplex Method: Maximization
(with problem constraints of the form $\leq$)

The graphical method works well for solving optimization problems with only two decision variables and relatively few constraints. However, it is unmanageable or impossible to use if there are more decision variables or many constraints.

To solve these, we will use an algebraic method called the *simplex method*, which was developed in 1947 by George Dantzig. Small problems can be done by hand, and computers can use the method to solve problems with thousands of variables and constraints.

Before using the simplex method, we will need to learn some new vocabulary and make some modifications to our mathematical models.

Example 1: This is the mathematical model used to solve and example from the previous section on the graphical method (the tables and chairs).

Maximize $P = 90x_1 + 25x_2$

subject to $8x_1 + 2x_2 \leq 400$.

$$2x_1 + x_2 \leq 120$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

This is an example of a *standard maximization problem*. It has a linear objective function along with constraints involving $\leq c$, where c is a positive constant. It also has nonnegative constraints for all the decision variables.

> Standard Maximization Problem in Standard Form
>
> A linear programming problem is said to be a *standard maximization problem in standard form* if its mathematical model is of the following form:
>
> Maximize $P = c_1x_1 + c_2x_2 + \ldots + c_nx_n$
>
> subject to problem constraints of the form
>
> $a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b$ with $b \geq 0$
>
> and nonnegative constraints $x_1, x_2, \ldots, x_n \geq 0$.

We will use the simplex method to solve standard maximization problems in standard form. The simplex method uses matrices to solve optimization problems. So, the constraint inequalities must be converted into equations before putting them into a matrix. This is done by the use of *slack variables*.

In our example,

$$\left.\begin{array}{l} 8x_1 + 2x_2 \le 400 \\ 2x_1 + x_2 \le 120 \end{array}\right\} \longrightarrow \begin{array}{l} 8x_1 + 2x_2 + s_1 = 400 \\ 2x_1 + x_2 + s_2 = 120 \end{array}$$

slack variables.

Note: if x_1 and x_2 satisfy the constraints, then $s_1 \ge 0$, $s_2 \ge 0$

These new variables s_1 and s_2 "take up the slack" between the left and right sides of our inequalities.

So now, our system looks like:

Max: $P = 90x_1 + 25x_2$

Subject to

$$\left.\begin{array}{l} 8x_1 + 2x_2 + s_1 = 400 \\ 2x_1 + x_2 + s_2 = 120 \\ x_1, x_2, s_1, s_2 \ge 0 \end{array}\right\}$$

our book calls this an e-system (system of equations)

Be sure to include the slack variables in the nonnegative constraints as well. If they are negative, the constraints are violated.

Next, rewrite the objective function with all variables on one side, making sure that the quantity *P* is positive.

$$-90x_1 - 25x_2 + P = 0$$

Including the new form of the objective function, we now have the *initial system*:

$$\begin{array}{l} 8x_1 + 2x_2 + s_1 = 400 \\ 2x_1 + x_2 + s_2 = 120 \\ -90x_1 - 25x_2 + P = 0 \\ x_1, x_2, s_1, s_2 \ge 0 \end{array}$$

We generate an augmented matrix for the initial system. This matrix is called the *initial simplex tableau*.

RHS = right-hand side

$$
\begin{array}{cccccc}
x_1 & x_2 & s_1 & s_2 & P & \text{RHS} \\
\end{array}
$$
$$
\left[\begin{array}{cccccc}
8 & 2 & 1 & 0 & 0 & 400 \\
2 & 1 & 0 & 1 & 0 & 120 \\
-90 & -25 & 0 & 0 & 1 & 0
\end{array}\right]
$$

$\underbrace{-90 \quad -25 \quad 0 \quad 0}_{\text{indicators}}$

The bottom row of the tableau always corresponds to the objective function. The numbers in the bottom row except for the two on the far right are called *indicators*. Above each column, we list each variable used in the system.

The Pivot:

To begin the simplex method, we perform a *pivot operation*.

First, we choose the *pivot column*. Our choice is determined by the indicators on the bottom row (the objective function row). If there are any negative numbers, we choose the "most negative" of these. The corresponding column is called the *pivot column*.

pivot element

$$
\begin{array}{cccccc}
x_1 & x_2 & s_1 & s_2 & P & \text{RHS} \\
\end{array}
$$
$$
\left[\begin{array}{cccccc}
8 & 2 & 1 & 0 & 0 & 400 \\
2 & 1 & 0 & 1 & 0 & 120 \\
-90 & -25 & 0 & 0 & 1 & 0
\end{array}\right]
\quad
\begin{array}{l}
\frac{400}{8} = 50 \leftarrow \text{choose the smallest} \\
\frac{120}{2} = 60
\end{array}
$$

most negative

Important: Do not form quotients from negative or zero values in the pivot column.

Next, we choose the *pivot row*. For each ~~nonnegative~~ positive entry in the pivot column, we divide the value in the rightmost by the corresponding ~~nonnegative~~ positive value in the pivot column. Do this in each row except for the bottom row.

Look for the smallest of these quotients that is positive. The row with the smallest quotient is called the *pivot row*. The element at the intersection of the pivot row and the pivot column is called the *pivot* or *pivot element*. We circle this element for easy recognition.

Pivot:
Use row operations to put a 1 in the position of the pivot element and 0's elsewhere in the pivot column.

To do this, we multiply the pivot row by the reciprocal of the pivot element. This will change the pivot element to a 1. Then we add multiples of the pivot row to all the other rows in order to change all other elements in the pivot column to 0.

Important Note: Never exchange rows when doing the simplex method!

want 1

$$\begin{bmatrix} 8 & 2 & 1 & 0 & 0 & 400 \\ 2 & 1 & 0 & 1 & 0 & 120 \\ -90 & -25 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\frac{1}{8}R_1 \to R_1$ $\sim$

$$\begin{bmatrix} 1 & 1/4 & 1/8 & 0 & 0 & 50 \\ 2 & 1 & 0 & 1 & 0 & 120 \\ -90 & -25 & 0 & 0 & 1 & 0 \end{bmatrix}$$

want 0's

$-2R_1 + R_2 \to R_2$ $\sim$
$90R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 1/4 & 1/8 & 0 & 0 & 50 \\ 0 & 1/2 & -1/4 & 1 & 0 & 20 \\ 0 & -5/2 & 45/4 & 0 & 1 & 4500 \end{bmatrix}$$

Scratch work

$$-2\,[1 \quad 1/4 \quad 1/8 \quad 0 \quad 0 \quad 50]$$
$$\Rightarrow [-2 \quad -1/2 \quad -1/4 \quad 0 \quad 0 \quad -100]$$
$$+R_2\,[2 \quad 1 \quad 0 \quad 1 \quad 0 \quad 120]$$
$$[0 \quad 1/2 \quad -1/4 \quad 1 \quad 0 \quad 20]$$

$$90R_1\,[90 \quad 90/4 \quad 90/8 \quad 0 \quad 0 \quad 4500]$$
$$+R_3\,[-90 \quad -25 \quad 0 \quad 0 \quad 1 \quad 0]$$
$$[0 \quad -10/4 \quad \tfrac{45}{4} \quad 0 \quad 1 \quad 4500]$$

This completes the pivot operation. We repeat the process, each time choosing a new pivot element, as long as there are negative numbers on the bottom row.

$$\begin{bmatrix} 1 & 1/4 & 1/8 & 0 & 0 & 50 \\ 0 & 1/2 & -1/4 & 1 & 0 & 20 \\ 0 & -5/2 & 45/4 & 0 & 1 & 4500 \end{bmatrix}$$

$\frac{50}{1/4} = 50(4) = 200$

$\frac{20}{1/2} = 20(\frac{2}{1}) = 40$ ← choose the smallest quotient

want 1

most negative

$2R_2 \to R_2$ $\sim$

$$\begin{bmatrix} 1 & 1/4 & 1/8 & 0 & 0 & 50 \\ 0 & 1 & -1/2 & 2 & 0 & 40 \\ 0 & -5/2 & 45/4 & 0 & 1 & 4500 \end{bmatrix}$$

want 0's

$-\frac{1}{4}R_2 + R_1 \to R_1$ $\sim$
$\frac{5}{2}R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 0 & 1/4 & -1/2 & 0 & 40 \\ 0 & 1 & -1/2 & 2 & 0 & 40 \\ 0 & 0 & 10 & 5 & 1 & 4600 \end{bmatrix}$$

next page

$$
\begin{array}{cccccc}
x_1 & x_2 & s_1 & s_2 & P & \text{RHS} \\
\end{array}
$$

$$
\left[\begin{array}{cccccc}
1 & 0 & 1/4 & -1/2 & 0 & 40 \\
0 & 1 & -1/2 & 2 & 0 & 40 \\
0 & 0 & 10 & 5 & 1 & 4600
\end{array}\right]
$$

No more negatives on bottom row, so we are at the maximum solution. (no more pivots are needed)

Basic variables: x_1, x_2, P (1 with 0s)

Nonbasic variables: s_1, s_2 (junk in columns)
(these are set equal to 0)

R_1: $1x_1 + 0x_2 + \frac{1}{4}s_1 - \frac{1}{2}s_2 + 0P = 40$

$1x_1 \quad + \frac{1}{4}(0) - \frac{1}{2}(0) = 40$

$x_1 + 0 + 0 = 40$

$x_1 = 40$

Since there are no negative numbers in the bottom row, we stop. This *final simplex tableau* represents the optimal solution.

The variables corresponding to the columns that look like columns of an identity matrix (a 1 in one entry and 0's elsewhere) are called *basic variables*. The variables corresponding to the other columns are called *nonbasic variables*. The solution represented by the simplex tableau is obtained by setting the nonbasic variables equal to 0.

variables with "junk in the columns" are nonbasic

These nonbasic variables are set equal to 0.

The solution is:

$$
\boxed{\begin{array}{l} x_1 = 40 \\ x_2 = 40 \\ P = 4600 \end{array}}
$$

Also note: $s_1 = 0$
$s_2 = 0$

So, we have obtained the same solution as we did using the geometric method.

<u>Notes</u>:

- If there are no positive numbers in the pivot column, then we cannot perform a pivot operation and we conclude that there is no optimal solution.

- If there is a tie for the "most negative" value in the bottom row, choose either column to be the pivot column. Unless there turns out to be no optimal solution, there will be multiple values of the variable that give the same maximum.

- If there is a tie for the smallest quotient when determining the pivot row, choose either row.

- The pivot element is always positive and is never in the bottom row.

Example 2: Use the simplex method to solve the following linear programming problem.

Maximize $P = 20x_1 + 10x_2$

subject to $x_1 + x_2 \le 5$.
$5x_1 + x_2 \le 9$
$x_1 \ge 0$
$x_2 \ge 0$

$$x_1 + x_2 + s_1 = 5$$
$$5x_1 + x_2 + s_2 = 9$$
$$-20x_1 - 10x_2 + P = 0$$

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \text{RHS} \end{array}$$
$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 5 \\ 5 & 1 & 0 & 1 & 0 & 9 \\ -20 & -10 & 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \frac{5}{1} = 5 \\ \frac{9}{5} = 1\frac{4}{5} \end{array}$$

most negative

$\frac{1}{5}R_2 \to R_2$

$\sim$

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 5 \\ 1 & 1/5 & 0 & 1/5 & 0 & 9/5 \\ -20 & -10 & 0 & 0 & 1 & 0 \end{array}\right]$$

want 0s

Choose smallest quotient

$-1R_2 + R_1 \to R_1$

$20R_2 + R_3 \to R_3$

$$\left[\begin{array}{cccccc} 0 & 4/5 & 1 & -1/5 & 0 & 16/5 \\ 1 & 1/5 & 0 & 1/5 & 0 & 9/5 \\ 0 & -6 & 0 & 4 & 1 & 36 \end{array}\right] \quad \begin{array}{l} \frac{16/5}{4/5} = \frac{16}{5} \cdot \frac{5}{4} = \frac{16}{4} = 4 \\ \frac{9/5}{1/5} = \frac{9}{5} \cdot \frac{5}{1} = \frac{9}{1} = 9 \end{array}$$

most negative

$\frac{5}{4}R_1 \to R_1$

$$\left[\begin{array}{cccccc} 0 & 1 & 5/4 & -1/4 & 0 & 4 \\ 1 & 1/5 & 0 & 1/5 & 0 & 9/5 \\ 0 & -6 & 0 & 4 & 1 & 36 \end{array}\right]$$

want 0s

$-\frac{1}{5}R_1 + R_2 \to R_2$

$\sim$

$6R_1 + R_3 \to R_3$

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \text{RHS} \end{array}$$
$$\left[\begin{array}{cccccc} 0 & 1 & 5/4 & -1/4 & 0 & 4 \\ 1 & 0 & -1/4 & 1/4 & 0 & 1 \\ 0 & 0 & 15/2 & 5/2 & 1 & 60 \end{array}\right]$$

No negatives on bottom row. We're done pivoting

Basic: x_1, x_2, P Nonbasic s_1, s_2

Note: $s_1 = 0$, $s_2 = 0$

$$\boxed{x_1 = 1, \; x_2 = 4, \; P = 60}$$

Example 3: Use the simplex method to solve the following linear programming problem.

Maximize $P = 2x_1 + 2x_2 + x_3$

subject to
$$\begin{aligned} 2x_1 + x_2 + 2x_3 &\le 14 \\ 2x_1 + 4x_2 + x_3 &\le 26 \\ x_1 + 2x_2 + 3x_3 &\le 28 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \\ x_3 &\ge 0 \end{aligned}$$

Example 4: Tristyn, who works for a non-profit organization, is raising money by visiting and telephoning local churches and businesses. She has discovered that each business requires a 2 hour personal visit and 1 hour of phone calls, while each church requires a 2 hour personal visit and 3 hours on the phone. Tristyn can typically raise \$1000 from each business and \$2000 from each church. In each month, she has 16 hours of time available for personal visits and 12 hours available for phone calls. Determine the most profitable mixture of groups she should contact and the most money she can raise in a month.

Example 5: The Sharp Company sells sets of kitchen knives. The Value Set consists of 2 paring knives and 1 medium knife. The Regular Set consists of 2 paring knives, 1 medium knife, and 1 chef's knife. The Deluxe Set consists of 3 paring knives, 1 medium knife, and 1 chef's knife. The profit is $30 on a Value Set, $40 on a Regular Set, and $60 on a Deluxe Set. The factory has on hand 800 paring knives, 400 medium knives, and 200 chef's knives. Assuming that all sets will be sold, how many of each type should be made up in order to maximize profit? What is the maximum profit?

Let x_1 = number of Value Sets

x_2 = number of Regular Sets

x_3 = number of Deluxe Sets

	Value	Reg	Deluxe	Maximum
Paring	2	2	3	800
Medium	1	1	1	400
Chef's	0	1	1	200
Profit	$30	$40	$60	—

Maximize $P = 30x_1 + 40x_2 + 60x_3$

Subject to:

$$2x_1 + 2x_2 + 3x_3 \le 800$$

$$x_1 + x_2 + x_3 \le 400$$

$$x_2 + x_3 \le 200$$

$$x_1, x_2, x_3 \geq 0$$

mathematical model

Example 6: A baker has 60 pounds of flour, 132 pounds of sugar, and 102 boxes of raisins. A batch of raisin bread requires 1 pound of flour, 1 pound of sugar, and 2 boxes of raisins, while a batch of raisin cakes needs 2 pounds of flour, 4 pounds of sugar, and 1 box of raisins. The profit is $30 for a batch of raisin bread and $40 for a batch of raisin cake. How many batches of each should be baked so that the profit is maximized? What is the maximum profit?