6.3: The Dual Problem: Minimization with $\geq$ Problem Constraints

The simplex method can be modified to solve minimization problems.

The transpose of a matrix:

The transpose of a matrix A is called A^T and is formed by interchanging the rows and columns of A.

Example 1: Find the transpose of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

$$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

The dual problem:

Every minimization problem with $\geq$ constraints can be associated with a maximization problem with $\leq$ constraints. This maximization problem is called the *dual problem*.

Example 1: Minimize $C = 2y_1 + y_2$

Subject to
$$\begin{aligned} y_1 + y_2 &\geq 8 \\ y_1 + 2y_2 &\geq 4 \\ y_1 &\geq 0 \\ y_2 &\geq 0 \end{aligned}$$

First, we create a matrix A using the constraints and the objective function, with the objective function on the bottom row:

$$A = \begin{bmatrix} 1 & 1 & 8 \\ 1 & 2 & 4 \\ 2 & 1 & * \end{bmatrix}$$

Next, form the transpose A^T:

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 8 & 4 & * \end{bmatrix}$$

(original variables were y_1, y_2)

From the transpose, write a new linear programming problem with $\boxed{\text{new variables:}}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 8 & 4 & * \end{bmatrix} \Rightarrow \quad \begin{aligned} x_1 + x_2 &\le 2 \\ x_1 + 2x_2 &\le 1 \\ 8x_1 + 4x_2 &= z \end{aligned}$$

choose a new variable for this also

The dual problem is:

$$\boxed{\begin{aligned} &\text{Maximize} && z = 8x_1 + 4x_2 \\ &\text{subject to} && x_1 + x_2 \le 2 \\ & && x_1 + 2x_2 \le 1 \\ & && x_1, x_2 \ge 0 \end{aligned}}$$

<u>Theorem of Duality</u>:

The objective function *w* of a minimizing linear programming problem takes on a minimum value if and only if the objective function *z* of the corresponding dual maximizing problem takes on a maximum value. The maximum value of *z* is equal to the minimum value of *w*.

So, after forming the dual problem, use the simplex method to solve it.

- For slack variables, use the variables of the original minimization problem.
- When writing the solution, the values of the original variables are read from the bottom row.

$$\begin{aligned} x_1 + x_2 + y_1 \quad &= 2 \\ x_1 + 2x_2 \quad + y_2 &= 1 \\ -8x_1 - 4x_2 \quad + z &= 0 \end{aligned}$$

$$\begin{array}{cccccc} x_1 & x_2 & y_1 & y_2 & z & RHS \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ -8 & -4 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{aligned} \tfrac{2}{1} &= 2 \\ \tfrac{1}{1} &= 1 \leftarrow \text{smallest quotient} \end{aligned}$$

most negative

$$\begin{array}{c} -1R_2 + R_1 \to R_1 \\ \sim \\ 8R_2 + R_3 \to R_3 \end{array} \quad \begin{array}{cccccc} x_1 & x_2 & y_1 & y_2 & z & RHS \end{array}$$

$$\begin{bmatrix} 0 & -1 & 1 & -1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 12 & 0 & 8 & 1 & 8 \end{bmatrix}$$

Solution to Dual Problem: Maximum is $z = 8$, when $x_1 = 1$, $x_2 = 0$

Solution to Original Problem: minimum is $C = 8$, when $y_1 = 0$, $y_2 = 8$

Example 2: (Example from Section 5.3—plant food)

Minimize $C = 30x_1 + 35x_2$

Subject to $20x_1 + 10x_2 \geq 460$

$$30x_1 + 30x_2 \geq 960$$
$$5x_1 + 10x_2 \geq 220$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Coefficient matrix A:

$$A = \begin{bmatrix} 20 & 10 & 460 \\ 30 & 30 & 960 \\ 5 & 10 & 220 \\ 30 & 35 & * \end{bmatrix} \qquad A^T = \left[\begin{array}{ccc|c} 20 & 30 & 5 & 30 \\ 10 & 30 & 10 & 35 \\ \hline 460 & 960 & 220 & * \end{array}\right]$$

The Dual Problem:

Maximize $P = 460y_1 + 960y_2 + 220y_3$

subject to

$$20y_1 + 30y_2 + 5y_3 \leq 30$$
$$10y_1 + 30y_2 + 10y_3 \leq 35$$
$$y_1, y_2, y_3 \geq 0$$

Write the system of equations:

$$20y_1 + 30y_2 + 5y_3 + x_1 = 30$$
$$10y_1 + 30y_2 + 10y_3 + x_2 = 35$$
$$-460y_1 - 960y_2 - 220y_3 + P = 0$$

Setup tableau and do all the pivots..

see next page

Final simplex tableau:

$$
\begin{array}{ccccccc}
y_1 & y_2 & y_3 & x_1 & x_2 & P & \\
\end{array}
$$

$$
\left[\begin{array}{ccccccc}
1 & 1 & 0 & 1/15 & -1/30 & 0 & 5/6 \\
-2 & 0 & 1 & -1/5 & 1/5 & 0 & 1 \\
60 & 0 & 0 & 20 & 12 & 1 & 1020
\end{array}\right]
$$

value of x_1 in original

value of x_2 in original

Solution to Dual Problem: $P = 1020$

$y_1 = 0$

$y_2 = 5/6$

$y_3 = 1$

Solution to Original Problem:

$$
\boxed{\begin{array}{c} \text{Minimum is } C = 1020 \\ x_1 = 20 \\ x_2 = 12 \end{array}}
$$

Example 3: An oil company operates two refineries in a certain city. Refinery I has an output of 200, 100, and 100 barrels of low-, medium-, and high-grade oil per day, respectively. Refinery II has an output of 100, 200, and 600 barrels of low-, medium-, and high-grade oil per day, respectively. The company wishes to produce at least 1000, 1400, and 3000 barrels of low-, medium-, and high-grade oil to fill an order. If it costs \$20,000/day to operate refinery I and \$30,000/day to operate refinery II, determine how many days each refinery should be operated to meet the requirements of the order at minimum cost to the company.