1324-BZBS14e-notes-8-3-condition-prob-independence

8.3: Conditional Probability, Intersection, and Independence

Conditional probability:

Consider the probability that a house will be flooded during a given year. Would you expect this probability to be different if you only considered houses that were located in a 50-year flood plain?

Example 1: Draw a single card from a standard 52-card deck.

a. What is the probability that you draw a jack?

S_1 = set of all 52 cards

$P(\text{Jack}) = \frac{4}{52} = \boxed{\frac{1}{13}}$

b. New information….Given that you drew a face card (K,Q,J), what is the probability that it is a jack?

$3(4) = 12$ face cards
4 of the 12 are jacks. So $P(\text{Jack given it's a face card})$ is $\frac{4}{12} = \boxed{\frac{1}{3}}$

new sample space: S_2 = set of face cards

Notation: $P(A \mid B)$ denotes the probability of A given that B occurs.

$P(A|B)$ = "probability of A given B" is "probability of the intersection divided by the probability of the given"

> Conditional probability definition:
>
> The probability of A given that B occurs is
>
> $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}. \qquad (P(B) \neq 0)$$

c. Use the conditional probability definition to determine the probability that a jack is drawn, given that the card is a face card.

Use original sample space: S_1 = set of 52 cards

J: A jack is drawn
F: A face card is drawn
Prob of the given: $P(F) = \frac{12}{52}$
$J \cap F$ = {Jacks of ♡, ♢, ♣, ♠}
Prob of the intersection: $P(J \cap F) = \frac{4}{52}$

$P(J|F) = \frac{P(J \cap F)}{P(F)} = \frac{4/52}{12/52}$

$= \frac{4}{\cancel{52}} \cdot \frac{\cancel{52}}{12} = \frac{4}{12} = \boxed{\frac{1}{3}}$

Example 2: Draw a single card from a standard 52-card deck. What is the probability of drawing the ace of diamonds given that the card is red?

One way: Stick with original sample space of 52 card and use formula:

$n(S) = 52$

A: Ace of diamonds
R: Red cards
$A \cap R$ = Ace of diamonds
$P(A \cap R) = \frac{1}{52}$
$P(R) = \frac{26}{52}$

$$P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{1/52}{26/52} = \frac{1}{\cancel{52}} \cdot \frac{\cancel{52}}{26} = \boxed{\frac{1}{26}}$$

2nd approach:

Change the sample space:
S_2 = set of red cards

$P(\text{Ace of Diamonds}) = \boxed{\frac{1}{26}}$

Example 3: When rolling a single die, what is the probability of rolling a prime given that the number rolled is even?

Using original sample space & cond prob formula

$S: \{1,2,3,4,5,6\}$
$\text{Primes} = \{2,3,5\}$
$\text{Evens} = \{2,4,6\}$
$\text{Primes} \cap \text{Evens} = \{2\} \Rightarrow P(\text{Prime} \cap \text{Even}) = \frac{1}{6}$

Note: 1 is in special category of its own & is neither prime nor composite

$$P(\text{Prime}|\text{Even}) = \frac{P(\text{Prime} \cap \text{Even})}{P(\text{Even})} = \frac{1/6}{3/6} = \boxed{\frac{1}{3}}$$

or

Redefining sample space to be the even numbers only (the given event)
$S_2 = \{2,4,6\}$
Which items in S_2 are prime? $\{2\}$
$n(S_2) = 3$
$P(\text{Prime}|\text{Even}) = \boxed{\frac{1}{3}}$

Example 4: In a test conducted by the U.S. Army, it was found that of 1000 new recruits, 680 men and 320 women, 57 of the men and 3 of the women were red-green color-blind. Given that a recruit selected at random from this group is red-green color-blind, what is the probability that the recruit is a male?

Ex 4½:

Year	Presidential Candidates	Women	Men	Gender Gap Percentage)	Source[2]
2016	Donald Trump (R)	41%	52%	11 pts.	Edison Research
	Hillary Clinton (D)	54%	41%		

(a) What is prob. a randomly chosen voter voted for Clinton, given the vote is a woman? randomly chosen voter voted for Trump, given prob

(b) What is prob a randomly chosen voter the voter is a woman?

(c) What is the prob. a randomly chosen voter voted for Clinton?

(a) $P(\text{Clinton}|W) = \boxed{\frac{54}{100}}$

(b) $P(\text{Trump}|W) = \boxed{\frac{41}{100}}$

(c) can't answer from this info... we can't assume equal numbers of men and women voters

Convert to raw numbers:

	W Women	M Men
Trump	41	52
Clinton	54	41
Other	5	7
Total	100	100

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	(W) Women	(M) Men	Total
Candidate A	5281	6947	12 228
Candidate B	4800	5211	10 011
TOTAL	10 081	12 158	22 239

(a) Prob a randomly chosen women voter voted for A?

(b) Prob a randomly chosen ~~man~~ men voter voted for A?

(c) Prob a randomly chosen voter voted for B?

(d) what percentage of B's votes are men?

(a) $P(A|W) = \frac{5281}{10\,081} \approx \boxed{0.524}$

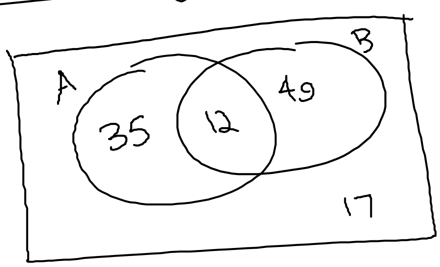
(b) $P(A|M) = \frac{6947}{12\,158} \approx \boxed{0.571}$

(c) $P(B) = \frac{10\,011}{22\,239} \approx \boxed{0.450}$

(d) $P(M|B) = \frac{5211}{10\,011} \approx \boxed{0.521}$

Conditional Probability with Venn diagram given:

$n(S) = n(U) = 35 + 12 + 49 + 17 = 113$



(a) What is $P(A|B)$?

$P(A|B) = \boxed{\frac{12}{61}}$ (redefining sample space to B only)

or use formula:

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{12/113}{61/113} = \boxed{\frac{12}{61}}$

The product rule for intersections of events:

Recall: The definition for conditional probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiply both side by $P(A)$: $P(B|A) \cdot P(A) = P(A \cap B)$

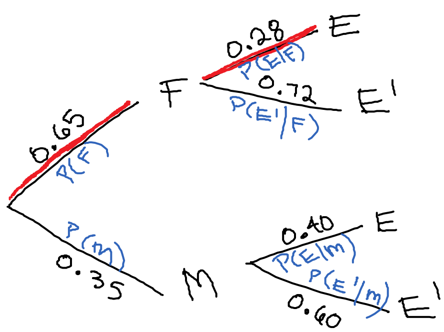
> Product rule:
>
> For events A and B with nonzero probabilities
>
> $P(A \cap B) = P(B \mid A)P(A)$

Example 5: In a certain class, 65% of the students are female. 40% of the males and 28% of the females are engineering majors.

F: Female, M = Male, E = Engineering Majors

a. What is the probability of a randomly selected student being female and an engineering major?

Probability Tree

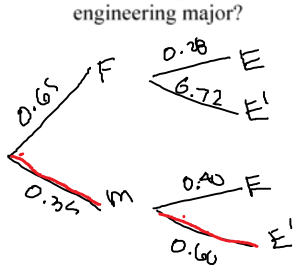


ⓐ $P(F \cap E) = P(F)P(E|F)$

$= 0.65\,(0.28)$

$= \boxed{0.182}$

b. What is the probability of a randomly selected student being male and a non-engineering major?

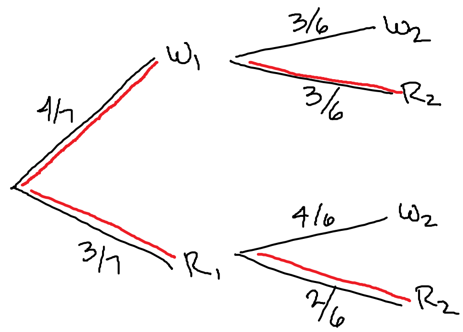


$P(M \cap E') = 0.35\,(0.60)$

$= \boxed{0.21}$

Example 6: A box contains 4 white marbles and 3 red marbles. Two marbles are drawn in succession without replacement. What is the probability of drawing a red marble on the second draw?

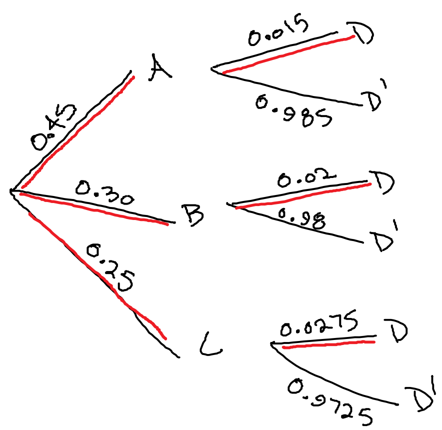
WWWW
RRR



$$P(R_2) = \frac{4}{7}\cdot\frac{3}{6} + \frac{3}{7}\cdot\frac{2}{6} = \frac{12}{42} + \frac{6}{42} = \frac{18}{42} = \boxed{\frac{3}{7}}$$

Example 7: A certain type of camera is manufactured in three locations. Plants A, B, and C supply 45%, 30%, and 25%, respectively, of the cameras. The quality-control department of the company has determined that 1.5% of the cameras produced by plant A, 2% of the cameras produced by plant B and 2.75% of the cameras produced by plant C are defective. What is the probability that a randomly selected camera is defective?

D: Defective
D': Nondefective (good)



$$P(D) = 0.45(0.015) + 0.30(0.02) + 0.25(0.0275) = \boxed{0.019625}$$

(So just under 2% of all cameras are defective)

Example 8: A box contains eight 9-volt transistor batteries, of which two are known to be defective. The batteries are selected one at a time without replacement and tested until a nondefective one is found. What is the probability that the number of batteries tested is

a. one?
b. two?
c. three?

Independence of events:

Example 9: (Compare with Example 6) A box contains 4 white marbles and 3 red marbles. Two marbles are drawn in succession, this time replacing the first before drawing the second.

a. What is the probability of drawing a red marble on the second draw?
b. What is the probability of drawing a red marble on the second draw given that a red marble was draw on the first draw?

Two events are said to be *independent* of the outcome of one does not affect the outcome of the other. If they are not independent, then they are said to be *dependent*.

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$P(A|B)P(B) = P(A\cap B)$$

If independent, can replace $P(A|B)$ by $P(A)$

$$P(A)P(B) = P(A\cap B)$$

<u>Independent Events</u>:
Events A and B are independent events if and only if:

- $P(A\,|\,B) = P(A)$ or, equivalently,
- $P(B\,|\,A) = P(B)$ or, equivalently,
- $P(A\cap B) = P(A)P(B)$.

<u>Example 10:</u> Draw a single card from a standard deck. Show whether the following pairs of events are independent or dependent.

a. Drawing a heart and drawing a face card.
b. Drawing a king and drawing a queen.

$n(F) = 3(4) = 12$

a) H: hearts
F: face cards

$P(H) = \frac{13}{52}$

$P(F) = \frac{12}{52}$

$H\cap F = \{Q\heartsuit, K\heartsuit, J\heartsuit\}$

$P(H\cap F) = \frac{3}{52}$

$P(H\cap F) \stackrel{?}{=} P(H)P(F)$

$\frac{3}{52} = \frac{13}{52}\cdot\frac{12}{52}$

$0.0577 \approx 0.0577$

$\Downarrow \quad \Downarrow$

$\frac{3}{52} \quad \frac{3}{52}$

$\boxed{P(H\cap F) = P(H)P(F) \text{ is true! So } H \text{ and } F \text{ are independent.}}$

b) $P(K) = \frac{4}{52}$

$P(Q) = \frac{4}{52}$

$P(K)P(Q) = \frac{4}{52}\cdot\frac{4}{52} = \frac{16}{2704}$

$= \frac{1}{169}$

Now $P(K\cap Q)$:

$K\cap Q = \emptyset$

$P(K\cap Q) = 0$

So, $P(K\cap Q) \neq P(K)P(Q)$, so K and Q are <u>dependent</u> (not independent)

<u>Example 11:</u> A survey conducted found that of 2000 women, 680 were heavy smokers and 50 had emphysema. Of those who had emphysema, 42 were also heavy smokers. Using this data, determine whether the events "being a heavy smoker" and "having emphysema" were independent events.

Independence of more than two events:

If $E_1, E_2, \ldots, E_n$ are independent, then

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1)P(E_2)\ldots P(E_n).$$

<u>Example 12:</u> A certain loudspeaker system has four components: a woofer, a midrange, a tweeter, and an electrical crossover. It has been determined that on the average 1% of the woofers, 0.8% of the midranges, 0.5% of the tweeters, and 1.5% of the crossovers are defective. Determine the probability that a randomly chosen loudspeaker is not defective. Assume that the defects in the different types of components are unrelated.