1324-BZBS14eNotes-7-4-perms-combs

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7.4: Permutations and Combinations

Permutations:

Example 1: Six horses are entered in a race. Assuming no ties, in how many possible ways can they finish first, second, and third?

$$\underset{1^{st}}{6} \cdot \underset{2^{nd}}{5} \cdot \underset{3^{rd}}{4} = \boxed{120}$$

This is a "permutation", or rearrangement.

Example 2: An Olympic event has ten competitors. In how many ways can the gold, silver, and bronze medals be awarded (assuming no ties)?

$$\underset{\text{Gold}}{10} \cdot \underset{\text{Silver}}{9} \cdot \underset{\text{Bronze}}{8} = \boxed{720}$$

Permutations of n Objects Taken r at a Time:

The number of permutations of n objects taken r at a time without repetition is given by

$$P_{n,r} = \frac{n!}{(n-r)!}$$

Note: $P_{n,n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$. So there are $n!$ ways to arrange n objects.

In Ex 2, $n = 10$, $r = 3$

There are $P_{10,3} = {}_{10}P_3 = \frac{10!}{(10-3)!}$

$= \frac{10!}{7!}$

$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}}$

$= 10 \cdot 9 \cdot 8 = \boxed{720}$

So for the last example, we have

Example: $P_{5,2} = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}} = 5 \cdot 4 = \boxed{20}$

$P_{7,7} = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1} = 7! = \boxed{5040}$

$\underset{1^{st}}{7} \cdot \underset{2^{nd}}{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 7!$

$P_{4,3} =$

Other notations: $P_{n,r}$ $\quad {}_nP_r$ $\quad P(n,r)$ $\quad P_r^n$

(all equivalent notations)

Combinations:

Example 3: A student group with ~~five officers~~ 10 members must form a three-member committee. How many different committees can be formed?

$\underline{10} \cdot \underline{9} \cdot \underline{8}$ however, this gives us too many possibilities we do not want different orderings/labellings to be counted as different outcomes

Need to divide by $3\cdot2\cdot1$

we end up with $\frac{10\cdot9\cdot8}{3\cdot2\cdot1} = \frac{720}{6} = \boxed{120 \text{ possibilities}}$

This is what we call a *combination* problem rather than a *permutation* problem.

Notice that in this situation, the order does not matter. In other words, "Joe, Mary, Sue" is the same committee as "Mary, Sue, Joe".

<u>Combinations of *n* Objects Taken *r* at a Time</u>:

The number of combinations of *n* objects taken *r* at a time without repetition is given by

$$C_{n,r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Note: $C_{n,n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{n!}{n!(1)} = 1.$

Back to our committees... We have $n=10$, $r=3$, so

$$C_{10,3} = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = \frac{10\cdot9\cdot8\cdot\cancel{7!}}{3\cdot2\cdot1\cdot\cancel{7!}} = \frac{720}{6} = \boxed{120}$$

or use nCr on your calculator

Example 4: $C_{6,2} = \boxed{15}$

$C_{7,4} = \boxed{35}$ $\quad C_{7,3} = \boxed{35}$

$C_{10,9} = \boxed{10}$ $\quad C_{10,1} = \boxed{10}$

Other notations: ${}_nC_r$ $\quad C_{n,r}$ $\quad C(n,r)$ $\quad C_r^n$ $\quad \binom{n}{r}$

Example 5: A student group with five members must choose a president, vice-president, and treasurer. In how many ways can this be done?

$P_{5,3} = \boxed{60}$

or $P_{5,3} = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\cdot4\cdot3\cdot\cancel{2\cdot1}}{\cancel{2\cdot1}} = 20\cdot3 = \boxed{60}$

OR $\underset{\text{Pres}}{\underline{5}} \cdot \underset{\text{VP}}{\underline{4}} \cdot \underset{\text{Treas}}{\underline{3}} = \boxed{60}$

Example 6: An art museum has a collection of 7 sculptures by a particular artist. There is only room to display four of the sculptures at a time. In how many different ways can four sculptures be chosen to display?

$C_{7,4} = \boxed{35}$

Very Important: • If order matters, use permutations. • If order does not matter, use combinations.

Example 7: Consider a standard 52-card deck.

a. How many 5-card hands contain 5 hearts?

There are 13 hearts, so $C_{13,5} = \boxed{1287}$

"13 choose 5"

How many possible 5-card hands?

$\boxed{C_{52,5} = 2\,598\,960}$

How many possible 6-card hands?

$\boxed{C_{52,6} = 20\,358\,520}$

b. How many 5-card hands will contain exactly 2 aces and 2 queens?

$\underbrace{C_{4,2}}_{2\text{ aces}} \cdot \underbrace{C_{4,2}}_{2\text{ Qs}} \cdot \underbrace{44}_{\text{1 non-ace non-queen}} = 6\cdot6\cdot44 = \boxed{1\,584}$

c. How many 5-card hands will contain 2 hearts and 3 clubs?

$\underbrace{C_{13,2}}_{2\text{ hearts}} \cdot \underbrace{C_{13,3}}_{3\text{ clubs}} = 78\cdot286 = \boxed{22\,308}$

If a 5-card hand is drawn randomly, what is the probability that it has 2 hearts and 3 clubs?

$\frac{C_{13,2}\cdot C_{13,3}}{C_{52,5}} = \frac{78\cdot286}{2\,598\,960} \approx \boxed{0.00858}$

Example 8: In how many ways can a quality-control engineer select a sample of 3 transistors for testing from a batch of 100 transistors?

$$C_{100,3} = \boxed{161\ 700}$$

Example 9: In how many different ways can a panel of 12 jurors and 2 alternate jurors be chosen from a group of 30 prospective jurors?

$$\underbrace{C_{30,12}}_{12\text{ jurors}} \cdot \underbrace{C_{18,2}}_{2\text{ alternates}} = 1.323346343 \times 10^{10} = \boxed{13,233,463,430}$$

OR $$\underbrace{C_{30,2}}_{2\text{ alternates}} \cdot \underbrace{C_{28,12}}_{12\text{ jurors}} = \text{Same answer!}$$

Example 10: In how many ways can a subcommittee of four be chosen from a Senate committee of five Democrats and four Republicans if

a. All members are eligible?

$$C_{9,4} = \boxed{126}$$

b. The subcommittee must consist of two Republicans and two Democrats?

$$\underbrace{C_{4,2}}_{2\text{ Republicans}} \cdot \underbrace{C_{5,2}}_{2\text{ Democrats}} = 6 \cdot 10 = \boxed{60}$$

play violin

play viola

plays cello

Example 11: The members of a string quartet composed of two violinists, a violist, and a cellist are to be selected from a group of six violinists, three violists, and two cellists.

a. In how many ways can the string quartet be formed?

$$\underbrace{C_{6,2}}_{2\text{ violins}} \cdot \underbrace{C_{3,1}}_{1\text{ viola}} \cdot \underbrace{C_{2,1}}_{1\text{ cello}} = C_{6,2} \cdot 3 \cdot 2 = 15 \cdot 3 \cdot 2 = \boxed{90}$$

b. In how many ways can the string quartet be formed if one of the violinists is to be designated as the first violinists and the other is to be designated as the second violinist?

$$\underbrace{6}_{1^{st}\text{ violin}} \cdot \underbrace{5}_{2^{nd}\text{ violin}} \cdot \underbrace{3}_{1\text{ viola}} \cdot \underbrace{2}_{\text{cello}} = \boxed{180}$$

(twice as many possibilities

OR $$\underbrace{P_{6,2}}_{2\text{ violins with order}} \cdot \underbrace{3}_{\text{viola}} \cdot \underbrace{2}_{\text{cello}} = \boxed{180}$$

Example 12: A student planning her curriculum for the upcoming year must select one of five Business courses, one of three Mathematics courses, two of six elective courses, and either one of four History courses or one of three Social Science courses. How many different curricula are available for her consideration?

$$\underbrace{5}_{\text{1 Business}} \cdot \underbrace{3}_{\text{1 math}} \cdot \underbrace{C_{6,2}}_{\text{2 electives}} \cdot \underbrace{7}_{\text{1 history or Social Science}} = 5 \cdot 3 \cdot 15 \cdot 7$$

$$= \boxed{1575}$$

Example 13: Twelve graduate students have applied for three available teaching assistantships. In how many ways can the assistantships be awarded among these applicants if

a. No preference is given to any student?

$$C_{12,3} = \boxed{220}$$

b. One particular student must be awarded an assistantship?

$$\underbrace{\underbrace{1}_{\text{1 special student}} \cdot \underbrace{C_{11,2}}_{\text{2 others}}}_{3} = \boxed{55}$$

c. The group of applicants includes seven men and five women and it is stipulated that at least one woman must be awarded an assistantship?

$$\underbrace{\underbrace{5}_{\text{1 woman}} \cdot \underbrace{C_{7,2}}_{\text{2 men}}}_{\text{3 assistantships}} + \underbrace{C_{5,2}}_{2W} \cdot \underbrace{7}_{1m} + \underbrace{C_{5,3}}_{\text{3 women}}$$

$$= 105 + 70 + 10$$

$$= \boxed{185}$$