1324-BZBS14e_Notes-3-2-compound-interest

Thursday, July 25, 2019 7:11 AM

1324-BZBS14e_Notes-3-2-compound-interest

3.2: Compound Interest

If at the end of a payment period, the interest due is reinvested at the same rate, then the interest as well as the principal will earn interest. This is called *compound interest*. The interest is paid into the account at the end of each compounding period.

4 times per year

Example 1: Suppose you invest $1000 compounded quarterly at an annual interest rate of 8%. How much money will you have after one year?

1st Quarter: (3 months)

at beginning, $P = \$1000$, $r = 0.08$, $t = \frac{1}{4}$ yr

$A = P(1+rt)$

$A = \$1000(1+0.08(\frac{1}{4}))$

$= \$1000(1.02) = \$1020$

2nd Qtr.

$P = \$1020$, $r = 0.08$, $t = \frac{1}{4}$ yr

$A = \$1020(1+0.08(\frac{1}{4}))$

$= \$1020(1.02)$

$= \$1040.40$

3rd Qtr:

$P = \$1040.40$, $r = 0.08$, $t = \frac{1}{4}$

$A = \$1040.40(1.02)$

$\approx \$1061.21$

4th Qtr. $P = \$1061.21$

$A = \$1061.21(1.02)$

$\approx \$1082.43$

You will have $1082.43 after 1 year.

Note: Q1: $\$1000(1.02)$

Q2: $\$1000(1.02)^2$

Q3: $\$1000(1.02)^3$

Q4: $\$1000(1.02)^4$

⋮

Q30: $\$1000(1.02)^{30}$

Compound Interest:

$$A = P(1+i)^n = P\left(1+\frac{r}{m}\right)^n = P\left(1+\frac{r}{m}\right)^{mt}$$

where

t = time in years

$i = \frac{r}{m}$ is the interest rate per compounding period

r = annual interest rate

m = number of compounding periods per year

n = total number of compounding periods ($n = mt$)

P = principal (present value)

A = amount (future value) at the end of n compounding periods.

In 3.2-3.4: assume a 365-day year

In 3.1: assume a 360-day year

annual compounding $m=1$
monthly $m=12$
semi annually $m=2$
daily $m=365$
quarterly $m=4$

Example 2: What is the future value of $1000 after 8 years at 6% compounded monthly?

→ so 12 times/yr

$A = P\left(1+\frac{r}{m}\right)^n$

$A = \$1000\left(1+\frac{0.06}{12}\right)^{96}$

$= \$1614.142708$

$\approx \$1614.14$

$\boxed{\text{Future value is } \$1614.14.}$

$A = ?$
$P = \$1000$
$r = 0.06$
$m = 12$
$n = mt = 12(8) = 96$
$t = 8$

Example 3: How much should I invest now at 4% interest compounded monthly in order to have $10,000 in 6 years?

$A = P\left(1+\frac{r}{m}\right)^n$

$\$10\,000 = P\left(1+\frac{0.04}{12}\right)^{72}$

$7869.418774 \approx P$

$\boxed{\text{You should invest } \$7869.42.}$

$A = \$10\,000$
$P = ?$
$r = 0.04$
$m = 12$
$n = mt = 12(6) = 72$
$t = 6$

Example 4: You decide to invest some money so that you will have $1,000,000 on your 75th birthday. At 8% compounded quarterly, how much should you invest on your 25th birthday?

$A = P\left(1+\frac{r}{m}\right)^n$

$\$1\,000\,000 = P\left(1+\frac{0.08}{4}\right)^{200}$

$P = \$19\,053.10$

$\boxed{\text{You should invest } \$19\,053.10.}$

$A = \$1\,000\,000$
$P = ?$
$r = 0.08$
$m = 4$
$n = mt = 4(50) = 200$
$t = 75-25 = 50$

n = total # of compounding periods (so $n = mt$)

m = # of compounding periods per year

Example 5: How long will it take \$5,000 to grow to \$7,000 if it is invested at 8% compounded monthly?

$A = P\left(1+\frac{r}{m}\right)^n$

$\$7000 = \$5000\left(1+\frac{0.08}{12}\right)^n$

$\frac{\$7000}{\$5000} = \left(1+\frac{0.08}{12}\right)^n$

$\frac{7}{5} = \left(1+\frac{0.08}{12}\right)^n$

$\ln\left(\frac{7}{5}\right) = \ln\left(1+\frac{0.08}{12}\right)^n$

$\ln\left(\frac{7}{5}\right) = n \cdot \ln\left(1+\frac{0.08}{12}\right)$

$\frac{\ln(7/5)}{\ln\left(1+\frac{0.08}{12}\right)} = n$

$A = \$7000$

$P = \$5000$

$r = 0.08$

$m = 12$

$n = mt = 12t$

$n \approx 50.6388853$ comp. periods

$n = mt$

$50.6388853 = 12t$

4.22 years $\approx t$

It needs to be invested for 51 compounding periods $\approx$ $\boxed{4.25 \text{ years}}$

$n = mt$

Recall: logarithms

$\log_2(8) = ?$ is asking $2^{?} = 8$

answer: 3 so $\log_2(8) = 3$

Property of logarithms:

$\log_b x^n = n \log_b x$

Also recall:

$\log_e(x) = \ln(x)$

The "natural log" of x

$e \approx 2.718$

Example 6: How long will it take money to double if it is invested at 7.5% compounded monthly?

$A = P\left(1+\frac{r}{m}\right)^n$

$2P = P\left(1+\frac{0.075}{12}\right)^n$

$\frac{2\cancel{P}}{\cancel{P}} = \frac{\cancel{P}\left(1+\frac{0.075}{12}\right)^n}{\cancel{P}}$

$2 = \left(1+\frac{0.075}{12}\right)^n$

$\ln(2) = \ln\left(1+\frac{0.075}{12}\right)^n$

$\ln(2) = n \cdot \ln\left(1+\frac{0.075}{12}\right)$

$\frac{\ln(2)}{\ln\left(1+\frac{0.075}{12}\right)} = n$

$n \approx 111.2498 \Rightarrow$ you need 112 compounding periods

$A = 2P$

$P = ?$ (but we don't care what P is)

$r = 0.075$

$m = 12$

$n = ?$

$n = mt$

$112 = 12t$

$\frac{112}{12} = t$

$\boxed{t \approx 9.33 \text{ years (9 years and 3 months)}}$

Continuous compound interest:

In calculus, a fundamental topic is the *limit*, or limiting value of a function. If we allow the number of compounding periods per year to increase toward infinity, the amount A approaches the limiting value $A = Pe^{rt}$. The number e is a constant, $e \approx 2.71828$. The number e is irrational—it cannot be written as a fraction of integers, or as a decimal that ends or repeats.

e can be defined as the limiting value of $\left(1+\frac{1}{x}\right)^x$ as x approaches ∞.

Start with the compound interest formula:

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

Substitute $x = \frac{m}{r}$ and then rearrange/simplify:

$$A = P\left[\left(1+\frac{1}{x}\right)^x\right]^{rt}$$

As $x \to \infty$, $\left(1+\frac{1}{x}\right)^x \to e$. This gives us the formula for continuous compound interest.

<u>Continuous Compound Interest</u>:

If principal P is compounded continuously at the annual interest rate r, then the amount at the end of t years is

$$A = Pe^{rt}.$$

<u>Example 7:</u> How much must be invested now to have $60,000 available in 10 years, if it is invested at 7% compounded (a) monthly? (b) continuously?

(a) monthly

$A = P\left(1+\frac{r}{m}\right)^n$

$n = mt$
$= 12(10) = 120$
$r = 0.07$
$m = 12$

$\$60\,000 = P\left(1+\frac{0.07}{12}\right)^{120}$

$\boxed{P = \$29\,855.78}$

(b) $A = Pe^{rt}$

$A = \$60\,000$
$P = ?$
$r = 0.07$
$t = 10$

$\$60\,000 = Pe^{0.07(10)}$

$\$60000 = Pe^{0.7}$

$\frac{\$60\,000}{e^{0.7}} = \frac{P\cancel{e^{0.7}}}{\cancel{e^{0.7}}}$

$P = \frac{\$60\,000}{e^{0.7}} \doteq \$29\,795.11823$

$\boxed{P \approx \$29\,795.12}$

Note: $\ln(e^x) = x$ Also $\ln(e) = 1$

also $e^{\ln x} = x$

because $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses

Example 8: How long will it take $5,000 to grow to $7,000 if it is invested at 8% compounded continuously?

$A = Pe^{rt}$

$A = \$7000$

$P = \$5000$

$t = ?$

$r = 0.08$

$7000 = 5000 e^{0.08t}$

$\frac{7000}{5000} = e^{0.08t}$

$\frac{7}{5} = e^{0.08t}$

$\ln\left(\frac{7}{5}\right) = \ln\left(e^{0.08t}\right)$ or $\ln\left(\frac{7}{5}\right) = 0.08t \cdot \ln(e)$

$\ln\left(\frac{7}{5}\right) = 0.08t$ $\ln\left(\frac{7}{5}\right) = 0.08t\ (1)$

$\frac{\ln(7/5)}{0.08} = t \Rightarrow \boxed{4.2059 \text{ years}}$

Effective rates:

The effective rate, sometimes called the *annual percentage yield*, converts a compound interest rate to an equivalent simple interest rate. This allows us to compare interest rates which have different compounding periods.

<u>Annual Percentage Yield (APY)</u>:

The annual percentage yield (APY), or effective rate, is given by

$$APY = r_e = \left(1 + \frac{r}{m}\right)^m - 1,$$

where

$r =$ annual interest rate

$m =$ number of compounding periods per year.

For interest compounded continuously, the APY is

$$APY = r_e = e^r - 1.$$

Example 9: What is the annual percentage yield for money invested at 6% compounded quarterly?

Example 10: Which investment is better, Note A at 9% compounded monthly or Note B at 9.2% compounded semiannually?