1324-BZBS14e_Notes-4-5-matrix-inverses

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4.5: Inverse of a Square Matrix

Identity matrix for multiplication:

Note: 0 is the additive identity

For real numbers, 1 is the identity for multiplication.

$1a = a \cdot 1 = a$ because $a+0=0+a=a$

Is there an identity for matrix multiplication? i.e. is there a matrix I such that $MI = IM = M$?

In general, there is no multiplicative identity that works, due to size incompatibility

However, for square matrices, there is such an identity.

For $n \times n$ matrices, I is the matrix with 1 on the principal diagonal and zeros elsewhere.

Examples of Identity Matrices:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2x 2

Note: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

also $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Inverses:

Every real number except 0 has a multiplicative inverse.

$a a^{-1} = a^{-1} a = 1$ ← multiplicative identity

Ex: For $a=7$, $a^{-1} = \frac{1}{7}$ because $7 \cdot \frac{1}{7} = \frac{1}{7} \cdot 7 = 1$

> The Inverse of a Square Matrix: (multiplicative inverse)
>
> Let M be an $n \times n$ square matrix and I be the $n \times n$ identity matrix. If there exists a matrix M^{-1} such that $M^{-1}M = MM^{-1} = I$, then M^{-1} is the inverse of M.

Example 1: Verify that $\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$ are inverses of one another.

$$\begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $\boxed{\text{Yes, these are inverses.}}$

How to find the inverse:

- To find the inverse of a matrix *M*, start by creating an augmented matrix $[M|I]$ by placing the appropriate-sized identity matrix to the right of the vertical line.
- Then row-reduce the augmented matrix until the identity matrix appears to the left of the vertical line. Then M^{-1} is to the right of the vertical line. In other words, row-reduce your augmented matrix until it looks like $[I|M^{-1}]$.
- If a zero row appears to the left of the vertical line, then M^{-1} does not exist.

Example 2: Find the inverse of $M = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, if it exists.

want 1

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_0 \longleftrightarrow R_1} \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right] \xrightarrow{-2R_0 + R_1 \rightarrow R_1}$$

want 0

2x4

want 0

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right] \xrightarrow{-1R_1 \rightarrow R_1} \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{-2R_1 + R_0 \rightarrow R_0}$$

want 1

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right] \Rightarrow \boxed{M^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}}$$

check it by multiplying

want: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & M^{-1} \\ 0 & 0 & 1 & \end{array}\right]$

Example 3: Find the inverse of $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, if it exists.

want 1

want 0s

$$\begin{array}{c} R_0 \\ R_1 \\ R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \underset{1R_0 + R_2 \to R_2}{\overset{-1R_0 + R_1 \to R_1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

want 0

want 1

want 0

$$\overset{-1R_1 + R_0 \to R_0}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \overset{1R_2 + R_0 \to R_0}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

want 0

$$\boxed{M^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$$

Example 4: Find the inverse of $M = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$, if it exists.

Example 5: Find the inverse of $M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$, if it exists.

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right] \sim^{\text{Row ops}} \left[\begin{array}{cc|cc} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

wait I cannot get it!

$\boxed{M^{-1} \text{ does not exist}}$

matrix M does not have an inverse

Shortcut for 2×2 **matrices:**

$D = \det(M) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the *determinant* of M is $D = ad - bc$. If $D \neq 0$, then M^{-1} exists and is given by

$$M^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

To get determinant

exchange places on principal diagonal

opp. signs on other diagonal

Example 6: Find the inverse of $M = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}$, if it exists.

$D = \det(M) = -5(4) - (-3)(6) = -20 - (-18) = -20 + 18 = -2$

$$M^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} \frac{4}{-2} & \frac{3}{-2} \\ \frac{-6}{-2} & \frac{-5}{-2} \end{bmatrix} = \boxed{\begin{bmatrix} -2 & -\frac{3}{2} \\ 3 & \frac{5}{2} \end{bmatrix} = M^{-1}}$$

Example 7: Find the inverse of $M = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$, if it exists.

$D = \det(M) = 2(6) - 4(3) = 12 - 12 = 0$

$\frac{1}{0}$ is not defined!

$\boxed{M^{-1} \text{ does not exist}}$

Example 8: Find the inverse of $M = \begin{bmatrix} 3 & 5 \\ 2 & 8 \end{bmatrix}$, if it exists.

$D = \det(M) = 3(8) - 5(2) = 24 - 10 = 14$

$M^{-1} = \frac{1}{14}\begin{bmatrix} 8 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{8}{14} & \frac{-5}{14} \\ \frac{-2}{14} & \frac{3}{14} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{4}{7} & -\frac{5}{14} \\ -\frac{1}{7} & \frac{3}{14} \end{bmatrix}}$