1314-1-5-Notes-Quadratic-Eqns-incl-complete-square

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1.5: Quadratic Equations

A *quadratic* equation in x is one that can be written in the general form $ax^2 + bx + c = 0$, where $a \neq 0$.

Quadratic equations can be solved by:
- factoring.
- taking square roots.
- completing the square
- using the quadratic formula.

Solving a quadratic equation by factoring:

Solving a quadratic equation by factoring involves the *zero-product principle*.

Zero-Product Principle
If the product of two algebraic expressions is zero, then at least one of the factors is equal to zero.
If $AB = 0$, then $A = 0$ or $B = 0$.

In other words, if multiplying two expressions results in 0, then at least one of them must be zero.

To solve an equation by factoring:

1. Move everything to one side (i.e. write in the form $ax^2 + bx + c = 0$).
2. Factor the nonzero side.
3. Set each factor equal to zero.
4. Solve each of the resulting new equations.

Example 1: Solve $x^2 - 3x = 4$ by factoring.

Example 2: Solve $3c^2 = 27c - 42$ by factoring.

Example 3: Solve $3A^2 - 10A = 8$.

$-8 \quad -8$ (−) signs are opposite – want a difference of 10 A for middle term

$3A^2 - 10A - 8 = 0$

$(3A + 2)(A - 4) = 0$

$3A + 2 = 0 \quad | \quad A - 4 = 0$

$-2 \quad -2 \quad | \quad +4 \quad +4$

$3A = -2 \quad | \quad A = 4$

$\frac{3A}{3} = \frac{-2}{3}$

$A = -\frac{2}{3}$

$3A^2 \quad 8$

$3A \cdot A \quad 1 \cdot 8$

$2 \cdot 4$

$2A$

$12A$

Check: $(3A+2)(A-4)$

$3A^2 - 12A + 2A - 8$

$3A^2 - 10A - 8$ ✓

Sol'n Set:

$\boxed{\left\{-\frac{2}{3}, 4\right\}}$

Example 4: Solve $x^2 = 25x$.

$-25x \quad -25x$

$x^2 - 25x = 0$

$x(x - 25) = 0$

$x = 0 \quad | \quad x - 25 = 0$

$x = 25$

Sol'n Set:

$\boxed{\{0, 25\}}$

Example 5: Solve $12t^2 + 5t - 2 = 0$.

Example 6: Solve $18t^2 + 25t + 4 = 0$.

The Square Root Property

If $u^2 = d$, then $u = \sqrt{d}$ or $u = -\sqrt{d}$.

Note: If $d > 0$, then both solutions are positive real numbers. If $d < 0$, then both solutions are non-real complex numbers.

Solving a quadratic equation by using the square root property:

1. Write it so that one side is a perfect square.
2. Take square roots of both sides, remembering the $\pm$. Both the positive and negative square roots make the equation true.

Example 7: Solve $x^2 = 100$ by factoring.

Get 1 side 0: $x^2 - 100 = 0$

$(x+10)(x-10) = 0$

$x + 10 = 0 \quad | \quad x - 10 = 0$

$x = -10 \quad | \quad x = 10$

Sol'n Set: $\boxed{\{-10, 10\}}$ or $\boxed{\{\pm 10\}}$

Example 8: Solve $x^2 = 100$ by taking square roots.

$x = \pm\sqrt{100}$

$x = \pm 10$

Sol'n Set: $\boxed{\{\pm 10\}}$

Note:
$\sqrt{a}$ is the **positive** number whose square is a. This means that $\sqrt{5}, \sqrt{6}, \sqrt{77},$ and $\sqrt{13}$ are all positive numbers.

Thus $-\sqrt{5}, -\sqrt{6}, -\sqrt{77},$ and $-\sqrt{13}$ are all negative numbers.

Question: What is $\sqrt{9}$?

$\sqrt{9} = \boxed{3}$

Example 9: Solve $y^2 - 17 = 0$ by using the square root property.

$+17 \quad +17$

$y^2 = 17$

$y = \pm\sqrt{17}$

Sol'n Set: $\boxed{\{\pm\sqrt{17}\}}$

or $\boxed{\{-\sqrt{17}, \sqrt{17}\}}$

Note: These solutions are approximately 4.1231 and −4.1231

Example 10: Solve $(w+3)^2 = 25$ by using the square root property.

$\sqrt{(w+3)^2} = \pm\sqrt{25}$

$w+3 = \pm\sqrt{25}$

$w+3 = \pm 5$

$w+3 = \pm 5$

$-3 \qquad -3$

$w = -3 \pm 5$

$w = -3+5$, or $w = -3-5$

$w = 2$ or $w = -8$

Sol'n Set: $\boxed{\{2, -8\}}$

Example 11: Solve $x^2 + 64 = 0$ by using the square root property.

$-64 \quad -64$

$x^2 = -64$

$x = \pm\sqrt{-64}$

$x = \pm 8i$

Sol'n Set: $\boxed{\{\pm 8i\}}$

Important: Any time you take square roots (or any even root) of both sides, you need the $\pm$.

Example 12: Solve $x^2 + 45 = 0$ by using the square root property.

$-45 \quad -45$

$x^2 = -45$

$x = \pm\sqrt{-45}$

$x = \pm i\sqrt{45}$

$x = \pm i\sqrt{9 \cdot 5}$

$x = \pm 3i\sqrt{5}$

Sol'n Set: $\boxed{\{\pm 3i\sqrt{5}\}}$

Solving a quadratic equation by completing the square:

To "complete the square on a quadratic equation:

1. Rewrite it with the constant term on the right side.
2. Divide both sides by the coefficient of x^2. You should now have something in the form $x^2 + bx = c$.
3. Add $\left(\frac{b}{2}\right)^2$ to both sides. This will make the left side into a perfect square.
4. Factor and solve by taking square roots.

Example 13: Solve $x^2 + 8x - 10 = 0$ by completing the square.

$+10 \quad +10$

$x^2 + 8x = 10$

$x^2 + 8x \qquad = 10$

$\left(\frac{8}{2}\right)^2 = (4)^2 = 16$

$x^2 + 8x + \underline{16} = 10 + \underline{16}$

$(x+4)(x+4) = 26$

$(x+4)^2 = 26$

$x + 4 = \pm\sqrt{26}$

$-4 \qquad -4$

$x = -4 \pm \sqrt{26}$

Sol'n Set: $\boxed{\{-4 \pm \sqrt{26}\}}$

Example 14: Solve $3x^2 - 18x + 1 = 0$ by completing the square.

$$3x^2 - 18x = -1$$
$$\frac{3x^2}{3} - \frac{18x}{3} = \frac{-1}{3}$$
$$x^2 - 6x = -\frac{1}{3}$$
$$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$
$$x^2 - 6x + 9 = -\frac{1}{3} + 9$$
$$(x - 3)^2 = -\frac{1}{3} + 9$$

[Divide both sides by 3 (coefficient of x^2)]

$$(x-3)^2 = -\frac{1}{3} + \frac{9}{1}\left(\frac{3}{3}\right)$$
$$(x-3)^2 = -\frac{1}{3} + \frac{27}{3}$$
$$(x-3)^2 = \frac{26}{3}$$
$$x - 3 = \pm\sqrt{\frac{26}{3}}$$
$$x - 3 = \pm\frac{\sqrt{26}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$
$$x = 3 \pm \frac{\sqrt{78}}{3}$$
$$x = \frac{3}{1}\left(\frac{3}{3}\right) \pm \frac{\sqrt{78}}{3}$$
$$x = \frac{9 \pm \sqrt{78}}{3}$$
$$\boxed{\left\{3 \pm \frac{\sqrt{78}}{3}\right\}} \text{ or } \boxed{\left\{\frac{9 \pm \sqrt{78}}{3}\right\}}$$

Example 15: Solve $x^2 + 7x - 5 = 0$ by completing the square.

$$x^2 + 7x = 5$$
$$\left(\frac{7}{2}\right)^2 = \frac{49}{4}$$
$$x^2 + 7x + \frac{49}{4} = 5 + \frac{49}{4}$$
$$\left(x + \frac{7}{2}\right)^2 = \frac{5}{1}\cdot\frac{4}{4} + \frac{49}{4}$$
$$\left(x + \frac{7}{2}\right)^2 = \frac{20}{4} + \frac{49}{4}$$
$$\left(x + \frac{7}{2}\right)^2 = \frac{69}{4}$$
$$x + \frac{7}{2} = \pm\sqrt{\frac{69}{4}}$$
$$x + \frac{7}{2} = \pm\frac{\sqrt{69}}{2}$$
$$x = -\frac{7}{2} \pm \frac{\sqrt{69}}{2}$$
$$\boxed{\left\{-\frac{7}{2} \pm \frac{\sqrt{69}}{2}\right\}} \text{ or } \boxed{\left\{\frac{-7 \pm \sqrt{69}}{2}\right\}}$$

Solving a quadratic equation using the quadratic formula:

Memorize this formula!

> The solutions to the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are given by
>
> $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This is called the <u>quadratic formula</u>.

Example 16: Solve $x^2 - 3x = 4$ using the quadratic formula.

$$x^2 - 3x - 4 = 0$$

[write in standard form $ax^2 + bx + c = 0$]

$a = 1$
$b = -3$
$c = -4$

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)} = \frac{3 \pm \sqrt{9 + 16}}{2}$$
$$= \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2}$$

$$x = \frac{3+5}{2} \qquad x = \frac{3-5}{2}$$
$$x = \frac{8}{2} \qquad x = \frac{-2}{2}$$
$$x = 4 \qquad x = -1$$

Sol'n Set:

$$\boxed{\{4, -1\}}$$

If you get solutions that are rational (fractions or integers), that tells you it could also be solved by factoring.

Example 17: Solve $t^2 + 3 = 5t$ using the quadratic formula.

$-5t \quad -5t$

$t^2 - 5t + 3 = 0$

$a = 1$
$b = -5$
$c = 3$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)} = \frac{5 \pm \sqrt{25 - 12}}{2}$$

$$= \frac{5 \pm \sqrt{13}}{2}$$

Sol'n Set: $\boxed{\left\{ \frac{5 \pm \sqrt{13}}{2} \right\}}$ or $\boxed{\left\{ \frac{5 + \sqrt{13}}{2}, \frac{5 - \sqrt{13}}{2} \right\}}$

Note: these solutions correspond to:
$x \approx 4.303$, $x \approx 0.697$

Example 18: Solve $3x^2 + 2x = -2$.

Example 19: Solve $u^2 - 6u + 9 = 0$.

Example 20: Solve $3x^2 - 4x = -1$.

Example 21: Solve $4y^2 - 3y = 10$.

Using the discriminant to determine the number of real solutions:

The *discriminant* is the expression under the square root sign in the quadratic formula.
(i.e. it=s the $b^2 - 4ac$.)

- If $b^2 - 4ac < 0$, the equation has no real solutions. There are two complex solutions, which are complex conjugates of one another.
- If $b^2 - 4ac > 0$, the equation has two real solutions.
- If $b^2 - 4ac = 0$, the equation has exactly one real solution.
- If $b^2 - 4ac$ is a perfect square (positive), and all the coefficients are rational, the equation has two rational solutions. (solutions are fractions or integers; no square roots)

Example 22: Describe the solutions of $4x^2 - 3x + 10 = 0$. ~~Solve it.~~

$a = 4$
$b = -3$
$c = 10$

$$b^2 - 4ac = (-3)^2 - 4(4)(10)$$
$$= 9 - 160$$
$$= -151$$

$b^2 - 4ac < 0 \Rightarrow$ negatives under square root

$\boxed{\text{2 non-real complex solutions}}$

Example 23: Describe the solutions of $5c^2 - 10c + 1 = 0$. ~~Solve it~~.

$b^2 - 4ac = (-10)^2 - 4(5)(1)$

$= 100 - 20$

$= 80$

$a = 5$
$b = -10$
$c = 1$

$\boxed{\text{2 real irrational solutions}}$

will have a square root in them

Example 24: Describe the solutions of $6x^2 + x = 12$. Solve it.

Example 25: Describe the solutions of $25a^2 - 30a + 9 = 0$. Solve it.