1.7: Linear Inequalities

An *inequality* is similar to an equation, except that in place of the equal sign, an inequality will have one of the symbols $<, >, \leq, \geq$. An inequality is called *linear* if each term is a constant or a multiple of the variable (i.e. it has no $x^2, \frac{1}{x}, \sqrt{x}$, etc.).

Ex:

Inequalities using the symbols <, > are called *strict* inequalities.

To *solve* an equation or inequality means to find all the values of the variable that make the equation or inequality true. Most of the equations we will solve have 1, 2, or maybe several solutions. Most inequalities have infinitely many solutions. Usually the solutions to an inequality form an interval or a union of intervals on the number line. We=ll only concern ourselves with real solutions to inequalities. (No complex numbers!)

Just as with a linear equation, we=ll solve a linear inequality by transforming the inequality into a series of *equivalent* inequalities by adding, multiplying, etc. the same thing to both sides.

In the following rules, the symbol $\Leftrightarrow$ means "is equivalent to".

Rules for inequalities: (these rules also apply to strict inequalities)

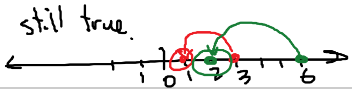
1. $A \leq B \quad \Leftrightarrow \quad A+C \leq B+C$.	**Adding** the same quantity to both sides does not change the inequality.
2. $A \leq B \quad \Leftrightarrow \quad A-C \leq B-C$.	**Subtracting** the same quantity from both sides does not change the inequality.
3. If $C > 0$, then $A \leq B \quad \Leftrightarrow \quad CA \leq CB$.	**Multiplying** (or dividing) both sides by the same **positive** quantity does not change the inequality.
4. If $C < 0$, then $A \leq B \quad \Leftrightarrow \quad CA \geq CB$.	**Multiplying** (or dividing) both sides by the same **negative** quantity **reverses** the inequality.
5. If $A > 0$ and $B > 0$, then $A \leq B \quad \Leftrightarrow \quad \frac{1}{A} \geq \frac{1}{B}$. $2 \leq 3$ $\frac{1}{2} \geq \frac{1}{3}$	Taking reciprocals of each side of an inequality involving **positive** quantities **reverses** the inequality.
6. If $A \leq B$ and $C \leq D$, then $A+C \leq B+D$.	Inequalities can be added.
7. If $A > 0$, then $\frac{1}{A} > 0$. If $A < 0$, then $\frac{1}{A} < 0$.	Taking reciprocals does not change the sign of a quantity.

Why must we reverse the inequality sign when we divide or multiply by a negative?

$3 < 6$ true

Divide by 3: $\frac{3}{3} < \frac{6}{3}$

$1 < 2$ still true

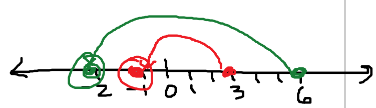


$3 < 6$ true

Divide by -3: $\frac{3}{-3} < \frac{6}{-3}$

$-1 < -2$ False!

To make it stay true, we need to reverse the inequality sign

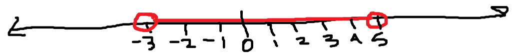


Example using numbers:

$-3 < x < 5$ means $-3 < x$ and $x < 5$
same as $x > -3$ and $x < 5$

Interval notation:

$(-3,5)$ All x such that $-3 < x < 5$

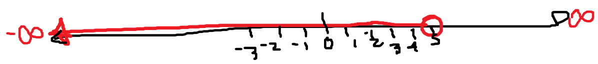


$[-3,5]$ All x such that $-3 \le x \le 5$

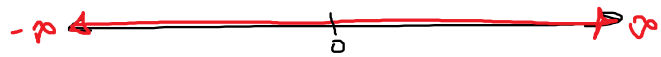
$[-3,5)$ All x such that $-3 \le x < 5$

$[-3,\infty)$ All x such that $x \ge -3$

$(-\infty,5)$ All x such that $x < 5$



$(-\infty,\infty)$ All real numbers



When solving inequalities, you should be able to write all answers in interval notation.

Example 1: Solve $4x+1 \le 7$.

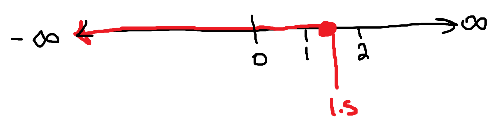
$-1 \quad -1$

$4x \le 6$

$\frac{4x}{4} \le \frac{6}{4}$

$x \le \frac{3}{2}$

$x \le 1\frac{1}{2}$ or $x \le 1.5$



Interval Notation $\boxed{(-\infty, 1\frac{1}{2}]}$ Sol'n Set

Set builder notation $\boxed{\{x \mid x \le 1\frac{1}{2}\}}$

Example 2: Solve $-2(6x+1) > 22$.

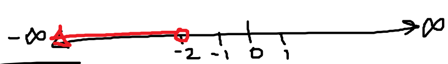
$-12x - 2 > 22$

$+2 \quad +2$

$-12x > 24$

$\frac{-12x}{-12} < \frac{24}{-12}$ ← Reverse the inequality sign!

$x < -2$



Solution Set in Interval Notation: $\boxed{(-\infty, -2)}$

Set builder: $\{x \mid x < -2\}$

Example 3: Solve $\frac{1}{5-2x} < 0$.

If the reciprocal of $5-2x$ is negative (is < 0), then $5-2x$ must be negative also (< 0)

$$5-2x < 0$$
$$+2x \quad +2x$$
$$5 < 2x$$
$$\frac{5}{2} < \frac{2x}{2}$$
$$\frac{5}{2} < x$$

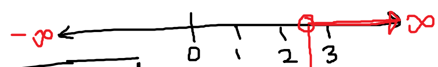
Rewrite with variable on left:

$$x > \frac{5}{2}$$

OR

$$5-2x < 0$$
$$-5 \quad -5$$
$$-2x < -5$$
$$\frac{-2x}{-2} > \frac{-5}{-2}$$
$$x > \frac{5}{2}$$
$$x > 2\tfrac{1}{2}$$

[Reverse the inequality sign!]



$2\frac{1}{2}$ write with the variable on the left

Interval Notation: $\boxed{(2\tfrac{1}{2}, \infty)}$

Set builder: $\{x \mid x > 2\tfrac{1}{2}\}$

Combined inequalities:

Sometimes, two inequalities can be written in a more compact way. This is called a *combined inequality* or a pair of *simultaneous inequalities*.

If you see $A < B \le C$, this means $A < B$ **and** $B \le C$. To combine two inequalities this way, both inequality signs must be going the same direction. **Never** write something like $A < B > C$.

Example 4: Solve $-3 \le 2z+1 < 7$.

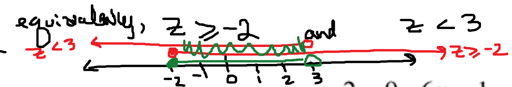
$$-3 \le 2z+1 < 7$$

We can separate them:

$$-3 \le 2z+1 \quad \text{and} \quad 2z+1 < 7$$
$$-1 \quad -1 \qquad\qquad -1 \quad -1$$
$$-4 \le 2z \qquad\qquad 2z < 6$$
$$\frac{-4}{2} \le \frac{2z}{2} \qquad\qquad \frac{2z}{2} < \frac{6}{2}$$
$$-2 \le z \qquad\qquad z < 3$$

equivalently, $z \ge -2$ and $z < 3$

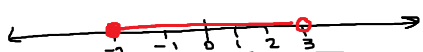
$\boxed{[-2,3)}$



OR

We can leave them combined:

$$-3 \le 2z+1 < 7$$
$$-1 \quad -1 \quad -1$$
$$-4 \le 2z < 6$$
$$\frac{-4}{2} \le \frac{2z}{2} < \frac{6}{2}$$
$$-2 \le z < 3$$



Interval notation: $\boxed{[-2,3)}$

Set builder: $\boxed{\{z \mid -2 \le z < 3\}}$

Example 5: Solve $-\frac{2}{3} < \frac{9-6x}{2} \le \frac{1}{6}$.

$$-\frac{2}{3} < \frac{9-6x}{2} \le \frac{1}{6}$$

Multiply all 3 sides by 6 to clear the fractions:

$$6\left(-\frac{2}{3}\right) < \left(\frac{9-6x}{\cancel{2}}\right)\cancel{6}^{3} \le \left(\frac{1}{6}\right)(6)$$
$$-\frac{12}{3} < 3(9-6x) \le \frac{6}{6}$$
$$-4 < 27-18x \le 1$$
$$-27 \quad -27 \quad -27$$
$$-31 < -18x \le -26$$
$$\frac{-31}{-18} > \frac{-18x}{-18} \ge \frac{-26}{-18}$$
$$\frac{31}{18} > x \ge \frac{26}{18}$$

[Reverse all the inequality signs]

Rewrite smallest to largest:

$$\frac{26}{18} \le x < \frac{31}{18}$$

<u>Example 6:</u> Solve $2 < 8 - 3x < -3$.

Absolute value inequalities:

To solve these we must remember that absolute value means **distance from zero!!!!**

<u>Example 1:</u> Solve $|x| \le 5$.

<u>Example 2:</u> Solve $|x| > 6$.

Summary: For a *positive* number *c*:

$$
\begin{aligned}
|x| < c \quad &\Leftrightarrow \quad -c < x < c \\
|x| \le c \quad &\Leftrightarrow \quad -c \le x \le c \\
|x| > c \quad &\Leftrightarrow \quad x < -c \text{ or } x > c \\
|x| \ge c \quad &\Leftrightarrow \quad x \le -c \text{ or } x \ge c
\end{aligned}
$$

<u>Example 3:</u> Solve $|-3x + 1| < 4$.

Example 4: Solve $2|1-4x|+1 \geq 7$.

Example 5: $13-3|4x-5| \leq 1$

Example 6: Solve $7+|2+3x| \leq 4$.

Example 7: Solve $|9x+2| \geq -5$.