2.1: Basics of Functions

Definition: A *relation* is any set of ordered pairs. The set of all first components of the ordered pairs is called the *domain* of the relation, and the set of all second components is called the *range* of the relation.

Example 1: $\{(1,5),(-3,6),(2,4),(1,6)\}$ What are the domain and range of this relation?

Domain: $\{1,-3,2\}$ Range: $\{5,6,4\}$

Definition: A *function* is a correspondence from a first set, called the domain, to a second set, called the range, such that each element in the domain corresponds to exactly one element in the range.

We can think of the domain as containing inputs and the range as containing outputs.

Definition: (informal) A *function* is a relation or rule in which every "input" is associated with exactly one "output".

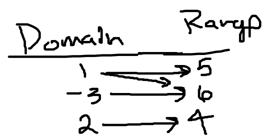
Another way to think about it: A function is like a machine. It takes an x (the input) and spits out exactly one y (the output).

Example 2: Are the following relations functions?

a. $\{(1,5),(-3,6),(2,4),(1,6)\}$

Not a function

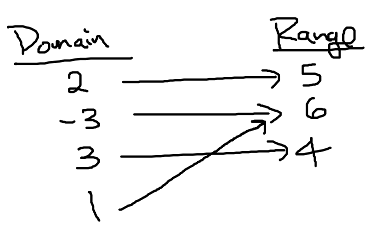
Input 1 gets sent to 2 different outputs



Can't have both $f(1)=5$ and $f(1)=6$

b. $\{(2,5),(-3,6),(3,4),(1,6)\}$

Yes, this is a function



Function Notation

$f(Math) = ☺$ [Read "f of $Math$ equals happy face"]

$Math$: the input (an element of the domain)

☺ : the output (an element of the range)

f : the name of the function (usually this is a letter, often f, g, or h.)

This notation is very helpful if we want our function to take real numbers as inputs. We can=t list them all in a table. For a function named *f*, $f(x)$ represents the value of the function at the number *x*.

x = 1st coordinate
y = 2nd coordinate

Example 1: Consider the relation $\{(3,5),(4,6),(-1,1),(1.5,3.5),(-7,-5)\}$. What equation describes the relationship?

$$y = x + 2$$

Notice that $y = 5$ when $x = 3$. Also $y = 6$ when $x = 4$. Writing $f(x)$ helps keep track of which *x* goes with which *y*.

We write: for $(3,5)$, we write $f(3) = 5$
for $(4,6)$ we write $f(4) = 6$
for $(-7,-5)$, write $f(-7) = -5$

Determining whether an equation defines a function:

Example 3: Does the equation $x^2 + 5y = 7$ define *y* as a function of *x*? (Can y be replaced by $f(x)$?)

Solve for y:

$$x^2 + 5y = 7$$
$$-x^2 \qquad -x^2$$
$$5y = -x^2 + 7$$
$$\frac{5y}{5} = \frac{-x^2+7}{5}$$
$$y = -\frac{1}{5}x^2 + \frac{7}{5} \quad \text{or} \quad y = \frac{-x^2+7}{5}$$

could write $f(x) = \frac{-x^2+7}{5}$ (just one answer for y)

So yes, the equation defines y as a function of x

Example 4: Does the equation $x^2 - 3y^2 = 6$ define *y* as a function of *x*?

Solve for y:

$$x^2 - 3y^2 = 6$$
$$-x^2 \qquad -x^2$$
$$-3y^2 = -x^2 + 6$$
$$\frac{-3y^2}{-3} = \frac{-x^2+6}{-3}$$
$$y^2 = \frac{-x^2}{-3} + \frac{6}{-3}$$
$$y^2 = \frac{1}{3}x^2 - 2$$
$$y = \pm\sqrt{\frac{1}{3}x^2 - 2}$$

2 answers for y, so $\boxed{y \text{ is not a function of } x}$

Example 5: Does the equation $x + y^3 = 7$ define *y* as a function of *x*?

Solve for y:

$$y^3 = -x + 7$$
$$y = \sqrt[3]{-x+7}$$

Only 1 answer for y, so $\boxed{\text{yes, } y \text{ is a function of } x}$

Could write $f(x) = \sqrt[3]{-x+7}$.

Earlier example: instead of $y = x+2$,

we write $f(x) = x+2$

$(3,5) \implies f(3) = 3+2 = 5$

Evaluating functions:

Example 2: Suppose $f(x) = 3x - 5$. Calculate $f(3)$.

$f(\boxed{}) = 3(\boxed{}) - 5$

$f(\boxed{3}) = 3(\boxed{3}) - 5$

$f(3) = 3(3) - 5$

$f(3) = 9 - 5$

$\boxed{f(3) = 4}$

Example 3: Suppose $g(x) = 3x^2 + 4x - 7$. Calculate $g(-2)$.

$g(\boxed{-2}) = 3(\boxed{-2})^2 + 4(\boxed{-2}) - 7$

$g(-2) = 3(-2)^2 + 4(-2) - 7$

$= 3(4) - 8 - 7 = 12 - 15 = -3$

$\boxed{g(-2) = -3}$

Example 4: Suppose $f(x) = -17$. Calculate $f(-2)$.

$f(\boxed{-2}) = -17$

$\boxed{f(-2) = -17}$

Note: $f(5) = -17$

$f\left(\frac{2317}{12}\right) = -17$

Domain

1, 2, 3, 8 $\to -17$

Example 5: Suppose $f(x) = 4x - 8$.

$f(\boxed{}) = 4(\boxed{}) - 8$

- Calculate $f(7x-2)$.

$f(\boxed{7x-2}) = 4(\boxed{7x-2}) - 8$

$f(7x-2) = 4(7x-2) - 8$

$= 28x - 8 - 8 = \boxed{28x - 16}$

- Calculate $f(a^2)$.

$f(\boxed{a^2}) = 4(\boxed{a^2}) - 8$

$\boxed{f(a^2) = 4a^2 - 8}$

Example 6: Suppose $g(x) = x^2 - 4x + 3$.

$g(\boxed{}) = (\boxed{})^2 - 4(\boxed{}) + 3$

- Calculate $g(-x)$.

$g(-x) = (-x)^2 - 4(-x) + 3$

$= \boxed{x^2 + 4x + 3}$ Could factor it, but no need.

Note: $-x^2 = -x^2$

$(-x)^2 = x^2$

- Calculate $g(x-2)$.

$g(x-2) = (x-2)^2 - 4(x-2) + 3$

$= (x-2)(x-2) - 4x + 8 + 3$

$= x^2 - 4x + 4 - 4x + 11$

$= \boxed{x^2 - 8x + 15}$

Example 7: Suppose $f(x)=\dfrac{2x-5}{7-x}$.

- Calculate $f(-3)$.

$$f(-3)=\frac{2(-3)-5}{7-(-3)}=\frac{-6-5}{7+3}=\frac{-11}{10}=\boxed{-\frac{11}{10}}$$

- Calculate $f(\frac{1}{3})$.

$$f\left(\tfrac{1}{3}\right)=\frac{2\left(\frac{1}{3}\right)-5}{7-\frac{1}{3}}=\frac{\frac{2}{3}-5}{7-\frac{1}{3}}\cdot\frac{3}{3}=\frac{2-15}{21-1}=\frac{-13}{20}=\boxed{-\frac{13}{20}}$$

- Calculate $f(x+h)$.

$$f(x+h)=\frac{2(x+h)-5}{7-(x+h)}=\boxed{\frac{2x+2h-5}{7-x-h}}$$

- Calculate $f\left(\dfrac{1}{x}\right)$.

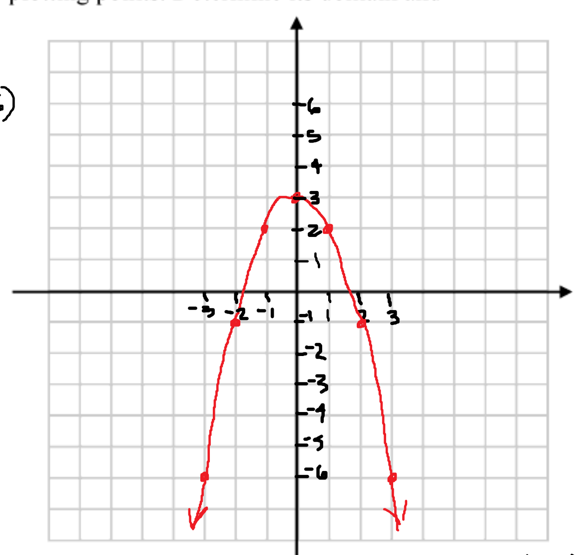
$$f\left(\tfrac{1}{x}\right)=\frac{2\left(\frac{1}{x}\right)-5}{7-\frac{1}{x}}\cdot\frac{x}{x}=\boxed{\frac{2-5x}{7x-1}}$$

Definition: The graph of a function *f* consists of those ordered pairs (x, y) such that x is in the domain of *f* and $y=f(x)$.

Graphing a function by plotting points:

Example 8: Sketch the graph of $f(x)=3-x^2$ by plotting points. Determine its domain and range from the graph.

x	$f(x)=3-x^2$
-3	$f(-3)=3-(-3)^2=3-9=-6 \Rightarrow (-3,-6)$
-2	$f(-2)=3-(-2)^2=3-4=-1 \Rightarrow (-2,-1)$
-1	$f(-1)=3-(-1)^2=3-1=2 \Rightarrow (-1,2)$
0	$f(0)=3-(0)^2=3-0=3 \Rightarrow (0,3)$
1	$f(1)=3-(1)^2=3-1=2 \Rightarrow (1,2)$
2	$f(2)=3-(2)^2=3-4=-1 \Rightarrow (2,-1)$
3	$f(3)=3-(3)^2=3-9=-6 \Rightarrow (3,-6)$



Domain: Set of valid inputs (x-values)

Range: Set of outputs (y-values)

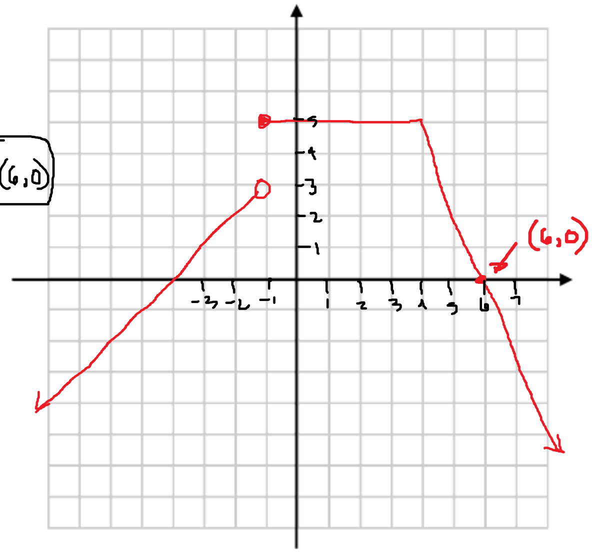
Domain: $(-\infty, \infty)$ (all real numbers)

Range: $(-\infty, 3]$ (same as $\{y \mid y \le 3\}$)

Plotting points is a useful way to graph functions, but it has limitations. You don't know for sure what the function is doing in between the points you plotted.

Understanding the graph of a function:

Example 9:



State the x- and y-intercepts.

$\boxed{x\text{-intercepts are } -4 \text{ and } 6}$, or $\boxed{(-4,0) \text{ and } (6,0)}$

$\boxed{y\text{-intercept is } 5 \text{ or } (0,5)}$

What is $f(-2)$? $\boxed{f(-2) = 2}$

What is $f(3)$? $\boxed{f(3) = 5}$

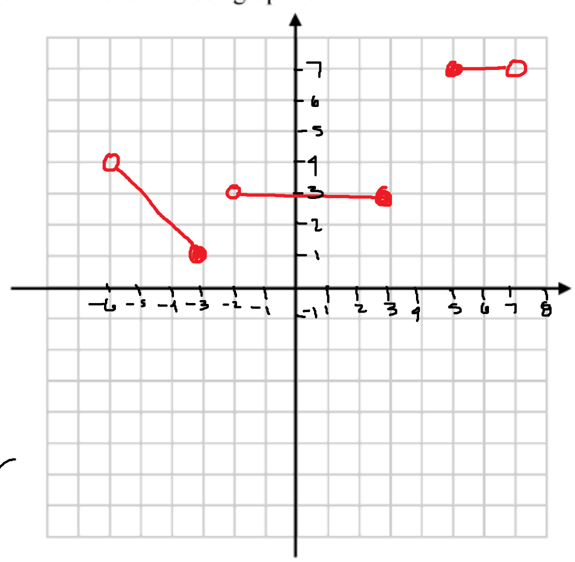
What is $f(0)$? $\boxed{f(0) = 5}$

What is $f(7)$? $\boxed{f(7) \approx -2\tfrac{1}{2}}$

Finding domain and range from a graph:

- Domain: the set of x-values that correspond to a point on the graph.
- Range: the set of y-values that correspond to points on the graph.

Example 10: Find the domain and range of the function whose graph is shown.



Domain: $(-6,-3] \cup (-2,3] \cup [5,7)$

Range: $[1,4) \cup \{7\}$

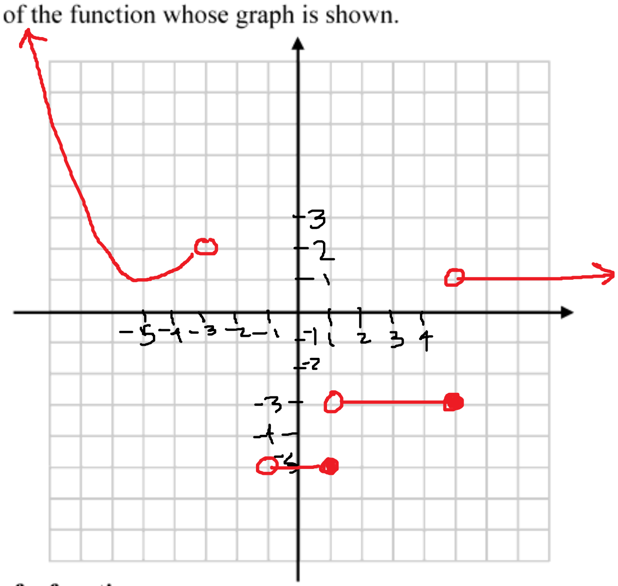
$\cup$: union symbol.
x must be in one set OR the other

$\cap$: Intersection symbol
x must be in 1st set AND in 2nd set

Example 11: Find the domain and range of the function whose graph is shown.

Domain: $(-\infty, -3) \cup (-1, \infty)$

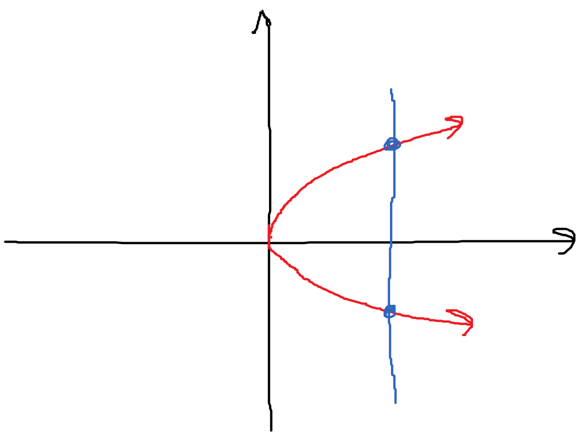
Range: $\{-5, -3\} \cup [1, \infty)$



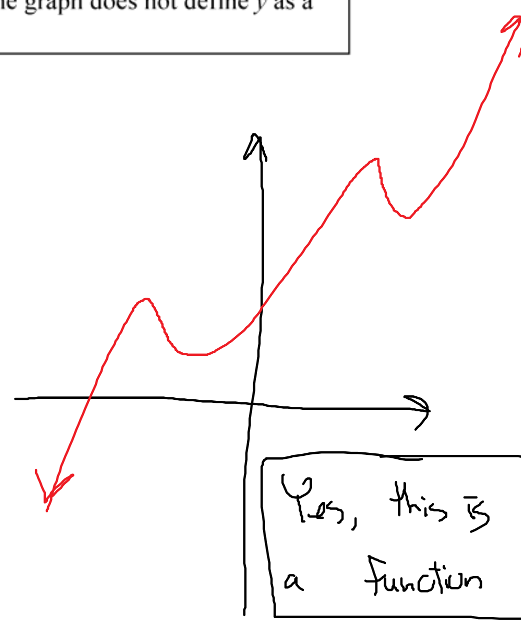
Recognizing when a graph is the graph of a function:

> Vertical Line Test:
>
> If any vertical line intersects a graph in more than one point, the graph does not define *y* as a function of *x*.

Example 12:



Not a funtion



Yes, this is a function