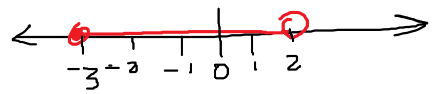
2.6: Combinations of Functions; Composite Functions

Domain of a function:

The *domain* is the set of all valid inputs for the function.

Sometimes the domain is stated specifically.

Example 1: $f(x) = 3x + 5; \quad -3 \le x < 2$



Here the domain is the interval [-3, 2).

Usually the domain of a function will *not* be specifically stated. We have to figure it out.

By convention, the domain is the set of all real numbers for which the expression is defined as a real number.

This means we=ll eliminate anything that results in zero denominators or even roots of negative numbers. Later, we=ll learn some more things that need to be eliminated.

$\cup$: union symbol

Example 2: State the domain of $f(x) = 3x - 4$.

$\boxed{\text{Domain is } (-\infty, \infty)}$ (All Real Numbers)

Example 3: State the domain of $h(x) = -17$.

Note: constant function
$h(3) = -17$
$h(5) = -17$
$h(-12) = -17$

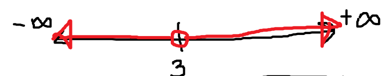
$\boxed{\text{Domain is } (-\infty, \infty)}$

Example 4: State the domain of $f(x) = \frac{5}{x-3}$.

Note: $\cancel{f(3) = \frac{5}{3-3}} \Rightarrow \frac{5}{0}$

Division by 0 is undefined $\Rightarrow$ throw out 3

$x - 3 \neq 0$
$x \neq 3$

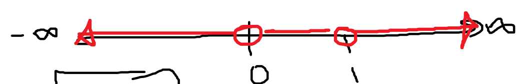


$\boxed{\text{Domain: } (-\infty, 3) \cup (3, \infty)}$

Example 5: State the domain of $g(x) = \frac{1}{x^2 - x}$.

Factor the denominator: $g(x) = \frac{1}{x(x-1)}$

$x \neq 0$, $x - 1 \neq 0$
$x \neq 1$



$\boxed{\text{Domain: } (-\infty, 0) \cup (0, 1) \cup (1, \infty)}$

Example 6: State the domain of $g(x) = \frac{1}{x^2 + 1}$.

$x^2 + 1 \neq 0$
$x^2 \neq -1$
$x \neq \pm\sqrt{-1}$
$x \neq \pm i$ not real

$x^2 + 1$ is never 0 for real numbers, so no need to throw anything out

$\boxed{\text{Domain is } (-\infty, \infty)}$

Note: x^2 + positive can never be 0, can never be negative

Recall: $\sqrt{25} = 5$

$\sqrt{7} = \sqrt{7}$, a positive number

$\left.\begin{array}{l}\sqrt{-25}\\ \sqrt{-7}\end{array}\right\}$ not real numbers

$\sqrt{0} = 0$

Solve: $x^2 = 7$

$x = \pm\sqrt{7}$

$x = \sqrt{7}, -\sqrt{7}$

Example 7: State the domain of $f(x) = \sqrt{5-2x} + 8$.

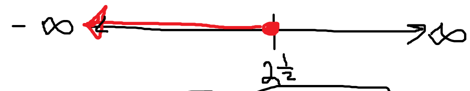
$5 - 2x \geq 0$

$-5 \qquad -5$

$-2x \geq -5$

Reverse the inequality sign! $\longrightarrow$ $\dfrac{-2x}{-2} \leq \dfrac{-5}{-2}$

$x \leq \frac{5}{2}$ or $x \leq 2\frac{1}{2}$



Domain: $\boxed{(-\infty, 2\tfrac{1}{2}]}$

or $\boxed{(-\infty, \tfrac{5}{2}]}$

Example 8: State the domain of $k(x) = \dfrac{x-9}{x^2 - x - 20}$.

$k(x) = \dfrac{x-9}{(x-5)(x+4)}$

$x - 5 \neq 0$ | $x + 4 \neq 0$

$x \neq 5$ | $x \neq -4$

Domain: $\boxed{(-\infty, -4) \cup (-4, 5) \cup (5, \infty)}$

Note: $k(9) = \dfrac{9-9}{(9-5)(9+4)} = \dfrac{0}{4(13)}$

$= \dfrac{0}{52} = 0$

9 gives us a valid output

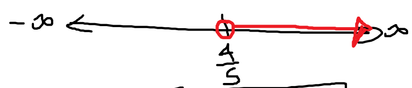
Example 9: State the domain of $g(x) = \dfrac{7-2x}{\sqrt{5x-4}}$.

$5x - 4 > 0$

$5x > 4$

$\dfrac{5x}{5} > \dfrac{4}{5}$

$x > \frac{4}{5}$



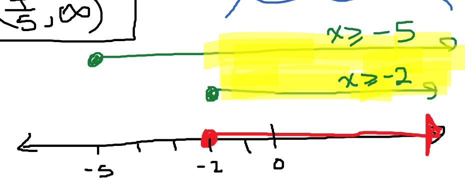
Domain: $\boxed{(\tfrac{4}{5}, \infty)}$

Example 10: State the domain of $f(x) = \sqrt{x+2} + \sqrt{x+5}$.

$x + 2 \geq 0$ and $x + 5 \geq 0$

$x \geq -2$ and $x \geq -5$

this area works in both



Domain of f: $\boxed{[-2, \infty)}$

Example 11: State the domain of $f(x) = \dfrac{\sqrt{4x+8}}{x-5}$.

$x - 5 \neq 0$

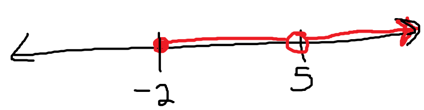
$x \neq 5$

also, $4x + 8 \geq 0$

$4x \geq -8$

$\dfrac{4x}{4} \geq \dfrac{-8}{4}$

$x \geq -2$



Domain: $\boxed{[-2, 5) \cup (5, \infty)}$

Recall: $\sqrt[3]{8} = 2$ because $(2)^3 = 8$

$\sqrt[3]{0} = 0$ because $(0)^3 = 0$

$\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$

$\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

$\sqrt[4]{-16}$ is not a real number because $(\ \ )^4 = -16$ is impossible with real numbers

Example 12: State the domain of $h(x) = \sqrt[5]{12x-7}$.

$\boxed{\text{Domain: } (-\infty, \infty)}$

Example 13: State the domain of $f(x) = \dfrac{\sqrt{6-2x}}{x^2-4}$.

Methods for combining functions:

1. Sum $\quad (f+g)(x) = f(x) + g(x)$
2. Difference $\quad (f-g)(x) = f(x) - g(x)$
3. Product $\quad (fg)(x) = f(x)g(x)$
4. Quotient $\quad \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ provided $g(x) \neq 0$.
5. Composition $\quad (f \circ g)(x) = f(g(x))$

Example 1: Suppose $f(x) = 2x-3$, $g(x) = x^2 - 4x + 5$, and $h(x) = 2x^3$.

a) $(f+g)(x) = f(x) + g(x)$

$= 2x - 3 + x^2 - 4x + 5$

$\boxed{(f+g)(x) = x^2 - 2x + 2}$

b) $(f+g)(-3)$

One way: Use $(f+g)(x) = x^2 - 2x + 2$

$(f+g)(-3) = (-3)^2 - 2(-3) + 2$

$= 9 + 6 + 2 = \boxed{17}$

2nd way: Find $f(-3)$ and $g(-3)$ separately & add them

$f(-3) = 2(-3) - 3$

$= -6 - 3$

$= -9$

$g(-3) = (-3)^2 - 4(-3) + 5$

$= 9 + 12 + 5$

$= 21 + 5 = 26$

$(f+g)(-3) = f(-3) + g(-3)$

$= -9 + 26$

$= \boxed{17}$

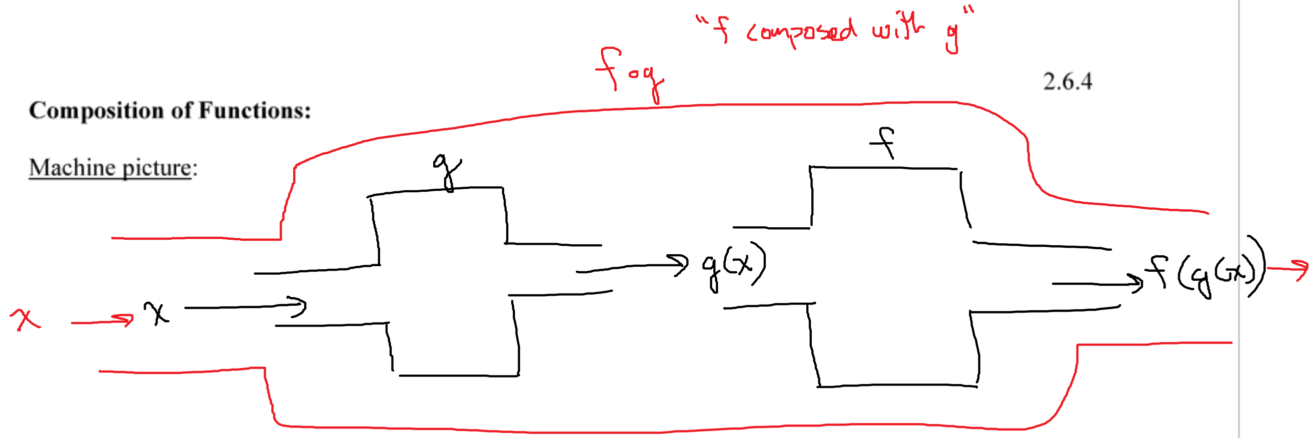
c) $(3f - g)(x)$

d) $\left(\dfrac{g}{h}\right)(x)$

$\left(\dfrac{g}{h}\right)(x) = \dfrac{g(x)}{h(x)} = \boxed{\dfrac{x^2 - 4x + 5}{2x^3}}$

Composition of Functions:

Machine picture:



This combined "machine" is called $f \circ g$ (read "f composed with g").

The new function $f \circ g$ is defined whenever x is in the domain of g and $g(x)$ is in the domain of f.

Scratchwork
$(2x-4)(2x-4)$
$= 4x^2 - 8x - 8x + 16 = 4x^2 - 16x + 16$

Example 2: Suppose $f(x) = 2x^2 - 3x + 1$ and $g(x) = 2x - 4$.

a) $(f \circ g)(x)$

$(f \circ g)(x) = f(g(x)) = f(2x-4) = 2(2x-4)^2 - 3(2x-4) + 1$
$= 2(4x^2 - 16x + 16) - 6x + 12 + 1$
$= 8x^2 - 32x + 32 - 6x + 13$
$= \boxed{8x^2 - 38x + 45 = (f \circ g)(x)}$

b) $(f \circ g)(-1)$

One way: use the eqn for $f \circ g$:
$(f \circ g)(x) = 8x^2 - 38x + 45$
$(f \circ g)(-1) = 8(-1)^2 - 38(-1) + 45$
$= 8(1) + 38 + 45 = 46 + 45 = \boxed{91}$

another way: calculate $g(-1)$ and put it into f:
$(f \circ g)(-1) = f(g(-1))$
$= f(-6)$
$= 2(-6)^2 - 3(-6) + 1$
$= 2(36) + 18 + 1$
$= 72 + 19 = \boxed{91}$

$g(-1) = 2(-1) - 4$
$= -2 - 4$
$= -6$

c) $(g \circ f)(x)$

$(g \circ f)(x) = g(f(x)) = g(2x^2 - 3x + 1)$
$= 2(2x^2 - 3x + 1) - 4$
$= 4x^2 - 6x + 2 - 4 = \boxed{4x^2 - 6x - 2}$

d) $(g \circ f)(-5)$

$(g \circ f)(-5) = 4(-5)^2 - 6(-5) - 2$
$= 4(25) + 30 - 2$
$= 100 + 28 = \boxed{128}$

e) $(g \circ g)(x)$

$(g \circ g)(x) = g(g(x)) = g(2x-4) = 2(2x-4) - 4$
$= 4x - 8 - 4$
$= \boxed{4x - 12}$

Example 3: Suppose $f(x)=\dfrac{3x-7}{4+8x}$, $g(x)=\dfrac{2}{x+4}$, and $h(x)=2x-5$.

a) $(f\circ h)(x)$

$$(f\circ h)(x) = f(h(x)) = f(2x-5) = \frac{3(2x-5)-7}{4+8(2x-5)} = \frac{6x-15-7}{4+16x-40} = \boxed{\frac{6x-22}{16x-36}} = \frac{\cancel{2}(3x-11)}{\cancel{2}(8x-18)} = \boxed{\frac{3x-11}{8x-18}}$$

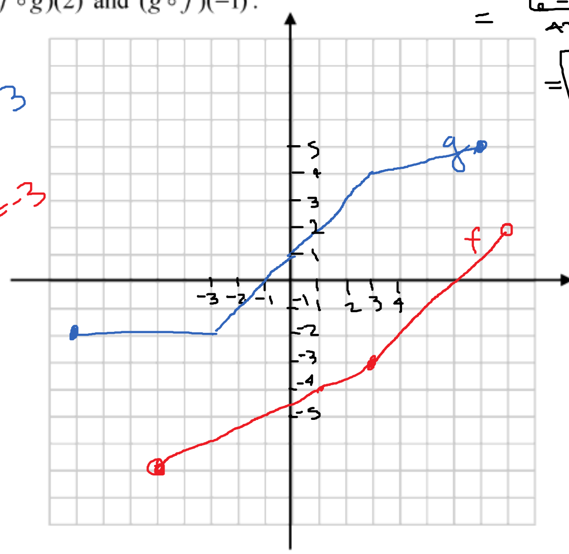
b) $(f\circ g)(x)$

$$(f\circ g)(x) = f(g(x)) = f\left(\frac{2}{x+4}\right) = \frac{\frac{3}{1}\left(\frac{2}{x+4}\right)-7}{4+\frac{8}{1}\left(\frac{2}{x+4}\right)} = \frac{\frac{6}{x+4}-7}{4+\frac{16}{x+4}}\cdot\frac{(x+4)}{(x+4)} = \frac{6-7(x+4)}{4(x+4)+16} = \frac{6-7x-28}{4x+16+16} = \boxed{\frac{-7x-22}{4x+32}}$$

Example 4: Use the given graph to determine $(f\circ g)(2)$ and $(g\circ f)(-1)$.

$$(f\circ g)(2) = f(g(2))$$
$$= f(3)$$ from graph, $g(2)=3$
$$= \boxed{-3}$$ from graph, $f(3)=-3$

$$(g\circ f)(-1) = g(f(-1))$$
$$= g(-5)$$
$$= \boxed{-2}$$



Finding the domain of compositions:

To find the domain of $f\circ g$, first compose them and find the rule for $f(g(x))$. Then find the domain of this new function $f(g(x))$. Also find the domain of $g(x)$ (the "inside" function). Combine these and you will have the domain of $f\circ g$.

Example 5: Suppose $f(x)=\dfrac{3x}{x-1}$ and $g(x)=x+5$.

a) Find $f\circ g$ and its domain.

$(f\circ g)(x) = f(g(x)) = f(x+5) = \dfrac{3(x+5)}{(x+5)-1} = \dfrac{3x+15}{x+5-1} = \boxed{\dfrac{3x+15}{x+4}}$

Domain of $f(g(x))$: $x \neq -4$

Domain of inside function $g(x)$: $g(x)=x+5$, has domain $(-\infty,\infty)$

Put these together: $x \neq -4$

b) Find $g\circ f$ and its domain.

$\boxed{\text{Domain of } f\circ g\text{: } (-\infty,-4)\cup(-4,\infty)}$

Example 5: Suppose $f(x)=\sqrt{x+5}$ and $g(x)=x^2$.

a) Find $f\circ g$ and its domain.

$(f\circ g)(x) = f(g(x)) = f(x^2) = \sqrt{(x^2)+5} = \boxed{\sqrt{x^2+5}}$

Domain of $f(g(x))$: $(-\infty,\infty)$

Domain of inside function $g(x)=x^2$: $(-\infty,\infty)$

$\boxed{\text{Domain of } f\circ g\text{: } (-\infty,\infty)}$

b) Find $g\circ f$ and its domain.

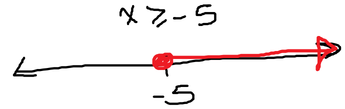
$(g\circ f)(x) = g(f(x)) = g(\sqrt{x+5}) = (\sqrt{x+5})^2 = \boxed{x+5} = (g\circ f)(x)$

Domain of combined function: $(-\infty,\infty)$

Domain of inside function $f(x)=\sqrt{x+5}$

$x+5 \geq 0$

$x \geq -5$



$\boxed{\text{Domain of } g\circ f\text{: } [-5,\infty)}$

Example 6: Suppose $f(x)=\frac{1}{x}$ and $g(x)=\sqrt{x+4}$. Find $f\circ g$ and its domain.

Finding domain of *f*+*g*, *f*-*g*, *fg*, *f*/*g*:

To be in the domain of these combinations of functions, x must be in the domain of both f and g.

Example 7: Suppose $f(x)=\sqrt{x+1}$ and $g(x)=\sqrt{x-5}$. Find the domain of $f+g$.