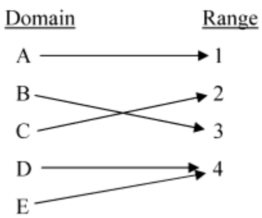
2.7

Thursday, October 31, 2019 8:55 PM

2.7: Inverse Functions

Definition: A function is *one-to-one* (1-1) if each member of the range is associated with exactly one member of the domain.

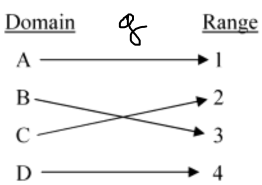
Example 1:



This function is $\boxed{\text{not one-to-one.}}$

output 4 in the range is associated with 2 inputs in the domain.

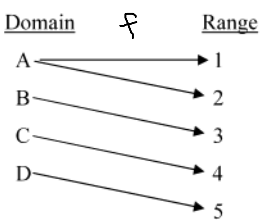
Example 2:



$\boxed{\text{This is a one-to one function}}$

$g(C) = 2, \quad g(D) = 4$

Example 3:



$f(A) = ?$

This relation is $\boxed{\text{not a function}}$.

Input A in the domain has 2 outputs in the range.

A one-to-one function has an *inverse* (another function that goes "backwards"). The inverse function reverses whatever the first function did.

Put another way, one function "does" something, the inverse function "undoes" it and you end up right where you started.

Note: $5^{-1} = \frac{1}{5^1} = \frac{1}{5}$

$x^{-1} = \frac{1}{x}$

The inverse of a function f is denoted by f^{-1}. Read this "*f*-inverse".(The inverse of a function g is denoted by g^{-1}, etc.)

> *Important: The -1 is not an exponent. !!*
>
> So $f^{-1}(x) \neq \dfrac{1}{f(x)}$!!!!!

Domain and Range: The range of f is the domain of f^{-1}. The domain of f is the range of f^{-1}.
When you think of inverses, think of exchanging the inputs and outputs, or "switching the x's and y's".

If the function is not one-to-one, you *cannot* do this!!

> These two statements mean *exactly* the same thing:
>
> 1. f is one-to-one.
> 2. f has an inverse function.

Example 4: $f(x) = 2x$

x	$f(x) = 2x$
-3	$2(-3) = -6$
-2	$2(-2) = -4$
-1	$2(-1) = -2$
0	$2(0) = 0$
1	$2(1) = 2$
2	$2(2) = 4$

x	$f^{-1}(x)$
-6	-3
-4	-2
-2	-1
0	0
2	1
4	2

In this case, $f^{-1}(x) = \frac{x}{2}$

How to tell if function are inverses of one another:

Two functions f and g are inverses if:

1. $f(g(x)) = x$ <u>and</u>
2. $g(f(x)) = x$

So, using our notation for *f*-inverse:

1. $f(f^{-1}(x)) = x$ <u>and</u>
2. $f^{-1}(f(x)) = x$

Example 5: Are $f(x)=5x+3$ and $g(x)=\frac{x-3}{5}$ inverses?

$f(g(x)) = f\left(\frac{x-3}{5}\right)$

$= \cancel{5}\left(\frac{x-3}{\cancel{5}}\right)+3$

$= x-3+3$

$= x$ ✓

$g(f(x)) = g(5x+3)$

$= \frac{(5x+3)-3}{5}$

$= \frac{5x+\cancel{3}-\cancel{3}}{5} = \frac{\cancel{5}x}{\cancel{5}} = x$ ✓

Example 6: Are $f(x)=\sqrt[3]{x-2}$ and $g(x)=2-x^3$ inverses?

Functions and their inverses:

In these examples, assume the functions are 1-1. Otherwise they won't have inverse functions!!

Example 7: If $f(7)=9$ and $f(9)=-12$, then what is $f^{-1}(9)$?

x	$f(x)$
7	9
9	−12

x	$f^{-1}(x)$
9	7
−12	9

$\boxed{f^{-1}(9) = 7}$

Example 8: If $f(4)=3$, what is $f(f^{-1}(3))$?

x	$f(x)$
4	3

$\Rightarrow$

x	$f^{-1}(x)$
3	4

$f^{-1}(3) = 4$

$f(f^{-1}(3)) = f(4) = \boxed{3}$

Example 9: Assume that both f and f^{-1} have the set of all real numbers as their domains.

If $f(-4)=7$ and $f(8)=10$, find $f^{-1}(f(2))$.

x	$f(x)$
−4	7
8	10
2	✱

$\Rightarrow$

x	$f^{-1}(x)$
7	−4
10	8
✱	2

$f^{-1}(f(2))$

$= f^{-1}(✱)$

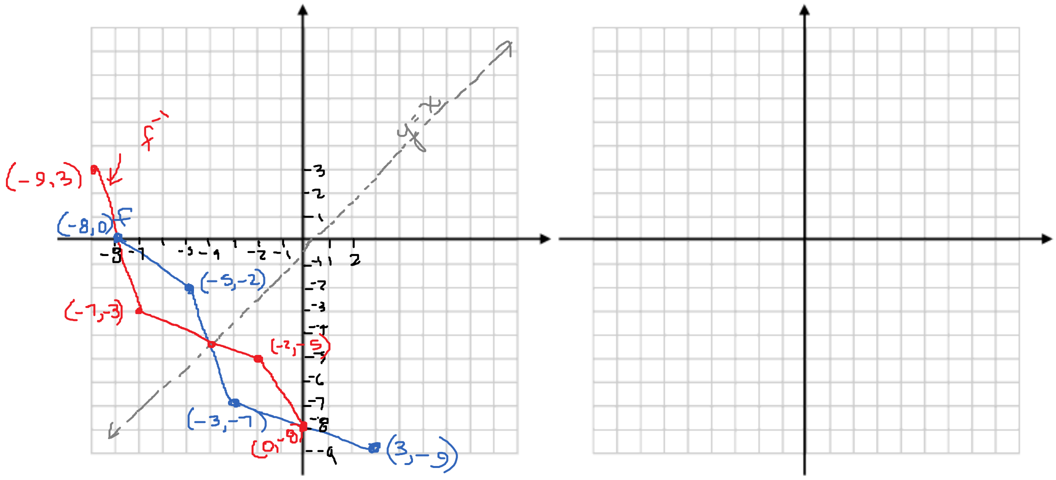
$= \boxed{2}$

Graphs of functions and their inverses:

The graphs of *f* and f^{-1} are symmetric about the line $y = x$.

$y = 1x + 0$ slope: $m = +\frac{1}{1}$, y-intercept 0 or (0,0)

Example 10: Given the graph of the function *f*, draw the graph of f^{-1}.

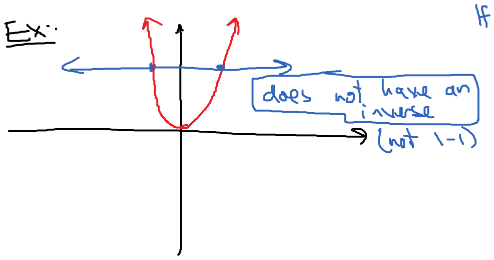


Using a graph to determine if a function has an inverse:

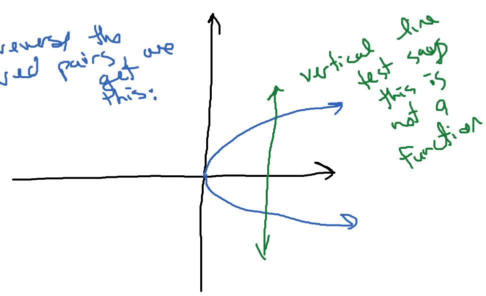
> Horizontal Line Test: A function is one-to-one (has an inverse) if and only if no horizontal line intersects its graph more than once.
>
> *So if you can draw a horizontal line that crosses it twice, then it does not have an inverse.*

Example 11:

Ex:



If we reverse the ordered pairs we get this:



Why does the horizontal line test work?

How to find the inverse of a function: (if it exists!)

1. Replace " $f(x)$ " by "y".
2. Exchange x and y.
3. Solve for y.
4. Replace "y" by " $f^{-1}(x)$ ".
5. Verify!!

Example 12: Find the inverse function of $g(x) = 2x - 7$. Don't forget to check your answer!

$y = 2x - 7$

Exchange x and y: $x = 2y - 7$
$+7 \quad +7$

Solve for y: $x + 7 = 2y$

$\frac{x+7}{2} = \frac{2y}{2}$

$\frac{x+7}{2} = y \Rightarrow \boxed{g^{-1}(x) = \frac{x+7}{2}}$

Check: $g(g^{-1}(x)) = g\left(\frac{x+7}{2}\right)$

$= \cancel{2}\left(\frac{x+7}{\cancel{2}}\right) - 7$

$= x + \cancel{7} - \cancel{7} = x$ ✓

Check the other way too!

Example 13: Find the inverse function of $f(x) = x^3 + 5$.

$y = x^3 + 5$

Switch x and y: $x = y^3 + 5$

Solve for y: $x - 5 = y^3$

$y^3 = x - 5$

$\sqrt[3]{y^3} = \sqrt[3]{x-5}$

$y = \sqrt[3]{x-5} \Rightarrow \boxed{f^{-1}(x) = \sqrt[3]{x-5}}$

Check our answer:

$f(f^{-1}(x)) = f(\sqrt[3]{x-5})$

$= (\sqrt[3]{x-5})^3 + 5$

$= x - \cancel{5} + \cancel{5}$

$= x$ ✓

$f^{-1}(f(x)) = f^{-1}(x^3 + 5)$

$= \sqrt[3]{(x^3+5) - 5}$

$= \sqrt[3]{x^3 + 5 - 5}$

$= \sqrt[3]{x^3} = x$ ✓

Example 14: Find the inverse function of $f(x) = (7-x)^3 + 1$.

Example 15: Find the inverse function of $g(x) = \frac{5}{x+3}$.

$y = \frac{5}{x+3}$

Switch x and y: $x = \frac{5}{y+3}$

$x(y+3) = 5$

$xy + 3x = 5$

$xy + 3x = 5$
$\quad\;\, -3x \quad -3x$

$xy = 5 - 3x$

$\frac{xy}{x} = \frac{5-3x}{x}$

$y = \frac{5-3x}{x} \Rightarrow \boxed{g^{-1}(x) = \frac{5-3x}{x}}$

Check: $g(g^{-1}(x)) = g\left(\frac{5-3x}{x}\right)$

$= \frac{5}{\left(\frac{5-3x}{x}\right)+3}$

$= \frac{5}{\left(\frac{5-3x}{x}\right)+3} \cdot \frac{(x)}{(x)}$

$= \frac{5x}{5-\cancel{3x}+\cancel{3x}}$

$= \frac{\cancel{5}x}{\cancel{5}} = x.$

Example 16: Find the inverse function of $h(x) = \frac{4x-2}{6+x}$.

$y = \frac{4x-2}{6+x}$

switch x and y: $x = \frac{4y-2}{6+y}$

$x(6+y) = 4y-2$

$6x + xy = 4y - 2$

Get all terms with y on 1 side; all terms without y on other side

$6x + xy = 4y - 2$

$xy - 4y = -2 - 6x$

$y(x-4) = -2 - 6x$

$\frac{y\cancel{(x-4)}}{\cancel{x-4}} = \frac{-2-6x}{x-4}$

$y = \frac{-2-6x}{x-4}$

$\boxed{h^{-1}(x) = \frac{-2-6x}{x-4}}$

Another way to write it: $h^{-1}(x) = \frac{-2-6x}{x-4}\left(\frac{-1}{-1}\right)$

$\boxed{h^{-1}(x) = \frac{2+6x}{-x+4}}$

$\boxed{h^{-1}(x) = \frac{6x+2}{4-x}}$

all are equivalent

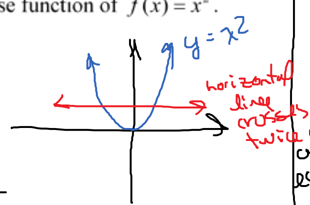
Example 17: Find the inverse function of $f(x) = x^2$.

$y = x^2$

Switch x and y: $x = y^2$

$y^2 = x$

Solve for y: $y = \pm\sqrt{x}$



2 answers for y!
It's not a function.

$\boxed{f(x) = x^2 \text{ does not have an inverse function}}$

(it's not 1-1)