3.2: Polynomial Functions and Their Graphs

Definition: A polynomial function is a function which can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 ,$$

n a positive integer

$a_0, a_1, a_2, \ldots, a_n$ are real numbers

Example 1:

Polynomials: $f(x) = 3x^4 - 7x^3 + 12x - \frac{3}{2}$ Exponents must be positive integers

$g(x) = 2 + x^2$, $h(x) = 5 - 2x - 7x^4$, $j(x) = \frac{1}{2}x^3 - \frac{7}{6}x^2 + \sqrt{3}\,x + \pi$

Not polynomials: $x^{\frac{3}{2}} + x^{\frac{1}{2}} = f(x)$

$g(x) = \frac{2}{x^3} - \frac{3}{x^2} + \frac{5}{x}$ (could rewrite as $g(x) = 2x^{-3} - 3x^{-2} + 5x^{-1}$)

$h(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[3]{x^2}$ (rewrite as $h(x) = x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{2}{3}}$)

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the coefficients of the polynomial function.

Note: The variable is only raised to positive integer powers–no negative or fractional exponents. However, the coefficients may be any real numbers, including fractions or irrational numbers like π or $\sqrt{7}$.

The degree of the polynomial is the largest exponent on x. (Degree is usually denoted by n.)

The leading coefficient of a polynomial is the coefficient of the term with the largest power of x.

Example 2: $f(x) = 2x^2 - 9x^4 + 7x^3 - 12$ Could rearrange: $f(x) = -9x^4 + 7x^3 + 2x^2 - 12$

Degree: 4

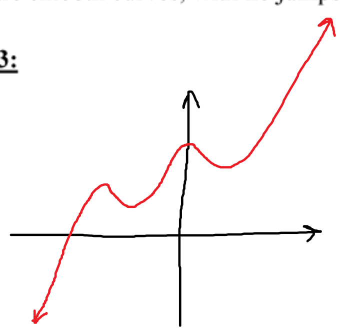
Leading coefficient: -9

Leading term: $-9x^4$

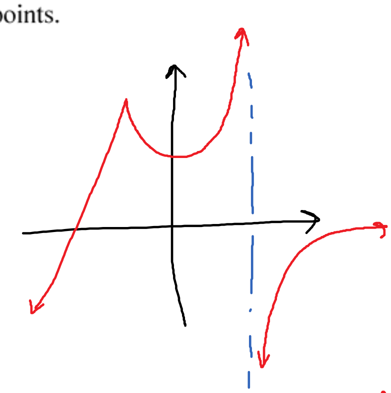
Facts about polynomials:

- They are smooth curves, with no jumps or sharp points.

Example 3:



Likely a polynomial

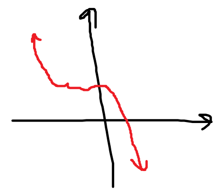


Not a polynomial

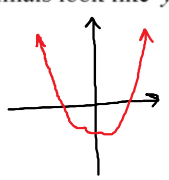
degree = n

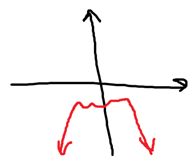
- A polynomial has at most $n-1$ turning points.
- A polynomial has at most n x-intercepts.
- A polynomial has exactly one y-intercept.
- Every polynomial has domain $(-\infty, \infty)$.
- Near the ends,

  Odd-degree polynomials look like $y = \pm x^3$.



  Even-degree polynomials look like $y = \pm x^2$.

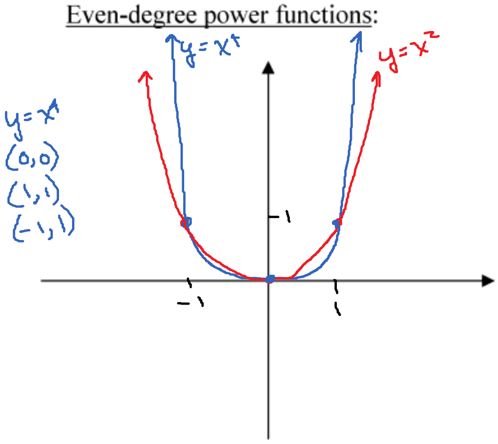




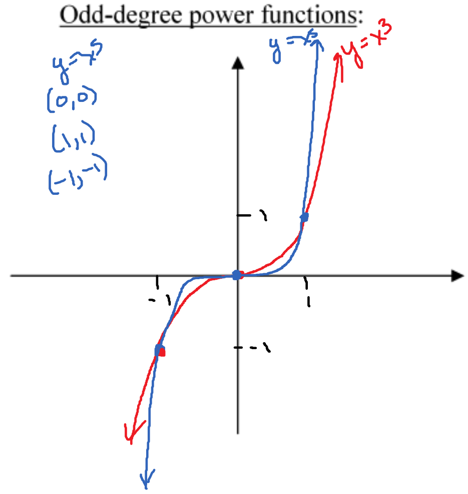
Power functions:

A *power function* is generally defined to be a polynomial which takes the form $f(x) = ax^n$, where n is a positive integer. Modifications of power functions can be graphed using transformations.

<u>Even-degree power functions</u>:



<u>Odd-degree power functions</u>:



<u>Note</u>: Multiplying any function by a will multiply all the y-values by a. The general shape will stay the same.

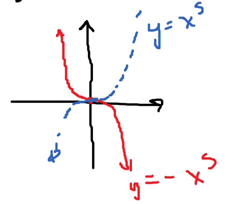
<u>Example 4:</u> Sketch the graph of $y = -(x+3)^5 - 4$.

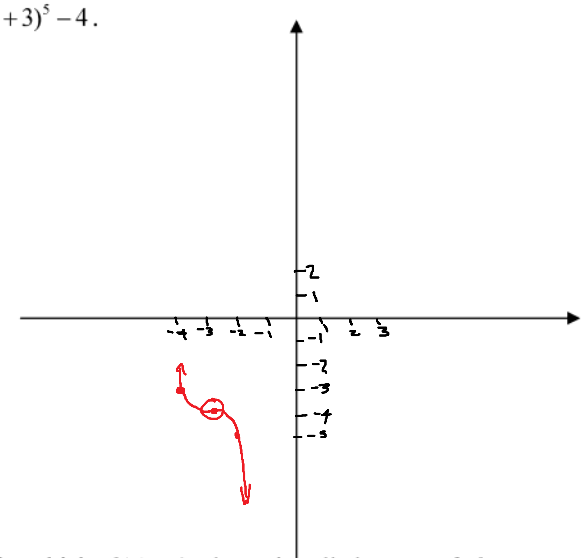
Parent function: $y = x^5$

Looks similar to x^3, but steeper.

Reflect around x-axis

shift it left 3, down 4





Zeros of polynomials:

If f is a polynomial and c is a real number for which $f(c) = 0$, then c is called a *zero* of f, or a *root* of f.

$f(x) = x^2 - 2x - 8$

Find x-intercepts: Set $y=0$

$0 = x^2 - 2x - 8$

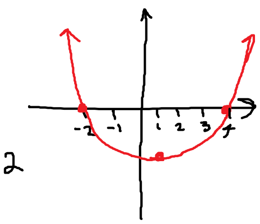
$0 = (x-4)(x+2)$

$x - 4 = 0 \quad | \quad x + 2 = 0$

$x = 4 \quad | \quad x = -2$

$\boxed{\text{Zeros: } 4, -2}$

Can rewrite: $f(x) = (x-4)(x+2)$



If c is a zero of f, then

- c is an x-intercept of the graph of f.
- $(x - c)$ is a factor of f.

So if we have a polynomial in factored form, we know all of its x-intercepts.

- every factor gives us an x-intercept.
- every x-intercept gives us a factor.

<u>Example 5:</u> Consider the function $f(x) = 3x(x-3)^6(2x-1)^3(x+2)^2$.

Zeros (x-intercepts):

$3x = 0 \quad | \quad x - 3 = 0 \quad | \quad 2x - 1 = 0 \quad | \quad x + 2 = 0$

$\frac{3x}{3} = \frac{0}{3} \quad | \quad x = 3 \quad | \quad 2x = 1 \quad | \quad x = -2$

$x = 0 \quad | \quad | \quad x = \frac{1}{2}$

$\boxed{\text{Zeros: } 0, 3, \tfrac{1}{2}, -2}$

To get the degree, add the multiplicities of all the factors.

exponents

$1 + 6 + 3 + 2 = 12$

$\boxed{\text{Degree} = 12}$

The leading term is: $\boxed{24x^{12}}$

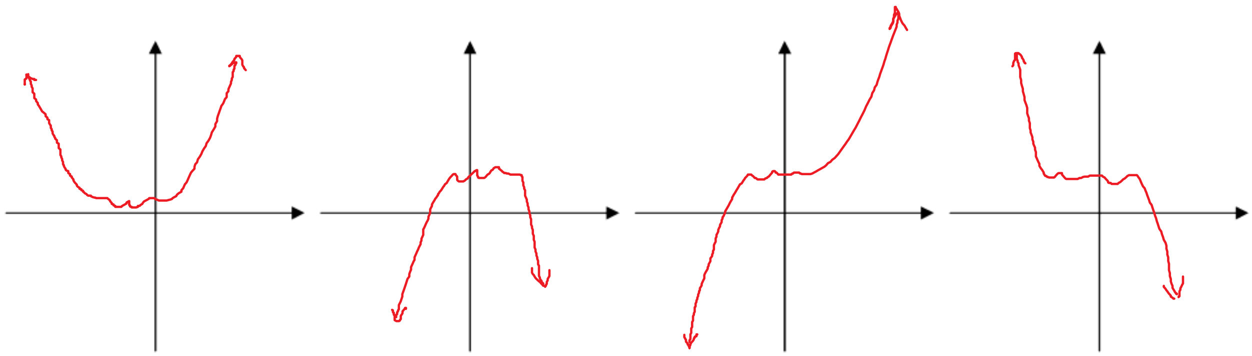
$3x\,(x)^6(2x)^3(x)^2 = 3x \cdot x^6 \cdot 2^3 x^3 \cdot x^2 = 3 \cdot x^7 \cdot 8x^3 \cdot x^2 = 24\,x^{12}$

Take leading term of each factor, raise it to the power on that factor, and multiply them together.

3.2.4

Steps to graphing other polynomials:

1. Factor and find x-intercepts.
2. Mark x-intercepts on x-axis.
3. Determine the leading term.
   - Degree: is it odd or even?
   - Sign: is the coefficient positive or negative?
4. Determine the end behavior. What does it "look like"?



Even Degree Sign (+)	Even Degree Sign (-)	Odd Degree Sign (+)	Odd Degree Sign (-)

5. For each x-intercept, determine the behavior.
   - Even multiplicity: touches x-axis, but doesn't cross (looks like a parabola there).
   - Odd multiplicity of 1: crosses the x-axis (looks like a line there).
   - Odd multiplicity ≥ 3 : crosses the x-axis and looks like a cubic there.

Note: It helps to make a table as shown in the examples below.

6. Draw the graph, being careful to make a nice smooth curve with no sharp corners.

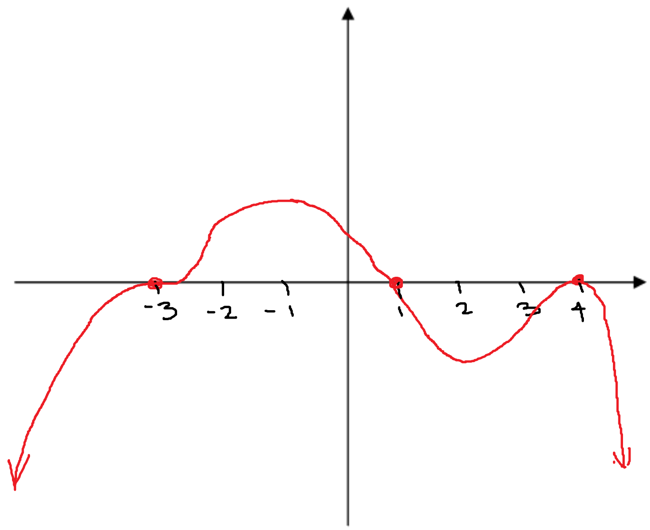
Note: without calculus or plotting lots of points, we don't have enough information to know how high or how low the turning points are.

Example 6: Sketch the graph of $g(x) = -(x-1)(x+3)^3(x-4)^2$.

Leading term: $-(x)(x)^3(x)^2$
$= -x^6$

Degree: 6 <u>even</u>
Sign: $-$
End Behavior:

Zeros	Multiplicity	Looks like
1	1	line
-3	3	cubic
4	2	parabola
	Sum: 6	



1 is a zero of multiplicity 1
-3 is a zero of multiplicity 3
4 is a zero of multiplicity 2

Example 7: Sketch the graph of $f(x) = x^3(4-x)(x+5)(x-8)^2$.

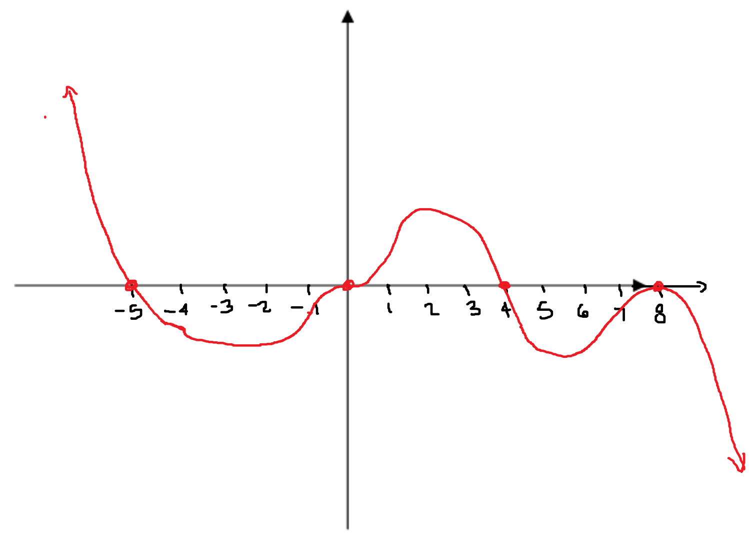
Leading term: $x^3(-x)(x)(x)^2$
$= -x^7$

Degree: 7 <u>odd</u>

Sign: $-$

End Behavior:

Zero	Multiplicity	Looks Like
0	3	cubic
4	1	line
-5	1	line
8	2	parabola
	$n=7$	

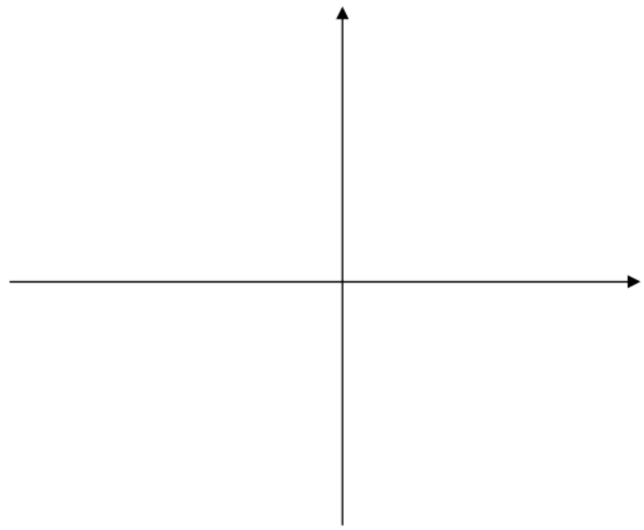


$x^3 = 0$
$x \cdot x \cdot x = 0$
$x = 0$

$4 - x = 0$
$4 = x$

Example 8: Sketch the graph of $y = 2(x+1)^5(x+7)^2(2x-7)$.

$y = 2(x+1)^5(x+7)^2(2x-7)$



Example 9: Sketch the graph of $P(x) = x^3 + 3x^2 - 4x - 12$.

$P(x) = x^3 + 3x^2 - 4x - 12$

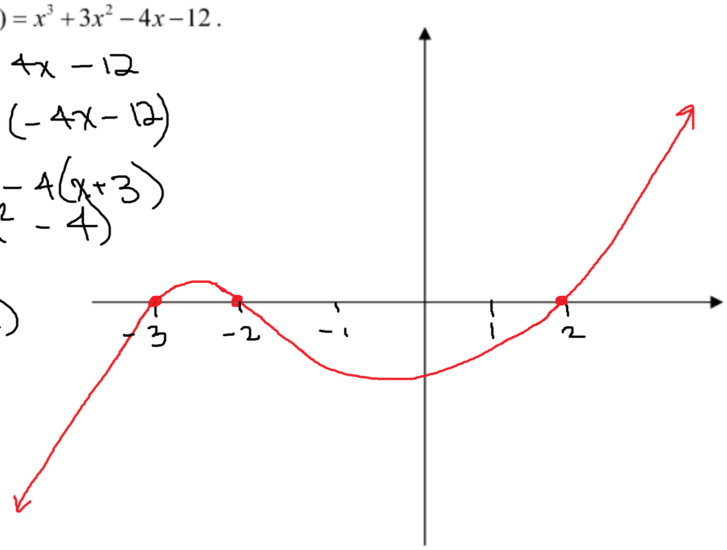
Factor it. Factor by grouping:

$P(x) = (x^3 + 3x^2) + (-4x - 12)$

$= x^2(x+3) - 4(x+3)$

$= (x+3)(x^2 - 4)$

$P(x) = (x+3)(x+2)(x-2)$



Leading Term: x^3

Degree: odd

Sign: +

End behavior:

x-intercepts	multiplicity	Looks Like
-3	1	line
-2	1	line
2	1	line

omit

Intermediate Value Theorem for Polynomials

Let f be a polynomial function with real coefficients. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of c between a and b for which $f(c)=0$.

Example 1: Show that $f(x)=3x^3-10x+9$ has a real zero between -3 and -2.